Received: 29 August 2022 | Revised: 30 January 2023 | Accepted: 3 February 2023

DOI: 10.1112/jlms.12732

RESEARCH ARTICLE

Journal of the London Mathematical Society

Strongly invertible knots, equivariant slice genera, and an equivariant algebraic concordance group

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Funding information
EPSRC, Grant/Award Numbers: EP/T028335/1, EP/V04821X/1

Abstract

We use the Blanchfield form to obtain a lower bound on the equivariant slice genus of a strongly invertible knot. For our main application, let K be a strongly invertible genus one slice knot with nontrivial Alexander polynomial. We show that the equivariant slice genus of an equivariant connected sum $\#^n K$ is at least $n/4$. We also formulate an equivariant algebraic concordance group, and show that the kernel of the forgetful map to the classical algebraic concordance group is infinite rank.

MSC 2020
57K10, 57N35, 57N70 (primary)

1 | INTRODUCTION

Let γ be a great circle in S^3, and let $\tau\colon S^3 \to S^3$ be the order two diffeomorphism given by the rotation with axis γ through π radians. Let K be a knot in S^3 that intersects γ in precisely two points, and such that $\tau(K) = K$. Then we say that K is *strongly invertible* with *strong inversion* τ. Note that $\tau|_K$ is necessarily orientation reversing, so K is in particular reversible. Suppose that K bounds a compact, oriented, locally flat surface Σ of genus g in D^4, such that for some extension of τ to a locally linear involution $\widehat{\tau}\colon D^4 \to D^4$, one has that $\widehat{\tau}(\Sigma) = \Sigma$. The minimal such g is called the (topological) *equivariant 4-genus* or *equivariant slice genus* of (K,τ), and denoted $\widetilde{g}_4(K,\tau)$. A strongly invertible knot with $\widetilde{g}_4(K,\tau) = 0$ is called *equivariantly slice*.

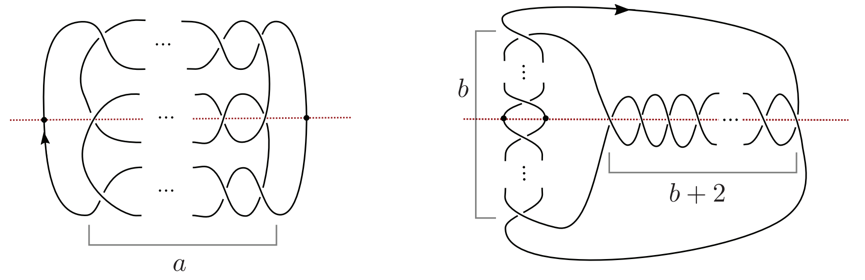

FIGURE 1 The pretzel knots $P(a, -a, a)$ for odd $a > 1$ (left) and the generalized twist knots K_b for even $b > 0$ (right) are strongly invertible genus one slice knots with nontrivial Alexander polynomials. The labels indicate the number of crossings in the given twist regions, with the illustrated handedness.

1.1 | Lower bounds on the equivariant 4-genus

By studying the Alexander module and the Blanchfield pairing, we derive a new lower bound for the equivariant 4-genus, which we will explain below. First, we state our main application.

Theorem 1.1. *Let K be a genus one algebraically slice knot with nontrivial Alexander polynomial and strong inversion τ. Let (K_n, τ_n) be an equivariant connected sum of n copies of (K, τ). Then the equivariant 4-genus of (K_n, τ_n) is at least $n/4$. In particular, if K is slice then $\widetilde{g}_4(K_n, \tau_n) - g_4(K_n) \to \infty$ as $n \to \infty$.*

There are many examples of strongly invertible genus one slice knots with nontrivial Alexander polynomial; see, for example, Figure 1. Thus, the topological equivariant 4-genus of strongly invertible slice knots can be arbitrarily large.

A fixed knot can admit multiple inequivalent strong inversions, as is the case for twist knots with even crossing number. The equivariant connected sum is also not unique, and depends on the choice of a direction [41] (see also Subsection 2.3). However, Theorem 1.1 holds for any choice of strong inversion on K and any equivariant connected sum K_n, so long as we use the same τ for each copy of K.

An involution $\tau : M^n \to M^n$ is *locally linear* if for every fixed point $x \in M$ there is a τ-invariant neighborhood U_x, a chart $\varphi : U_x \to \mathbb{R}^n$ around x with $\varphi(x) = 0$, and there is a linear transformation $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi \circ \tau = \psi \circ \varphi$. Every smooth involution is locally linear, and local linearity is the natural analogue of smoothness in the topological category, introduced in [4]. See [42] for a survey of work on locally linear actions.

The study of strongly invertible knots up to equivariant concordance was instigated by Sakuma [41], who defined an equivariant knot concordance group of directed strongly invertible knots, and introduced the η-polynomial, a homomorphism from the equivariant knot concordance group to the abelian group $\mathbb{Z}[t^{\pm 1}]$. The η-polynomial was originally defined as an obstruction to smooth equivariant concordance. However, using results from [22] one can show that it extends to an obstruction in the topological category. Borodzik–Dai–Mallick–Stoffregen inform us that a proof of this will appear soon in work of theirs on equivariant concordance, so to avoid duplicating effort we will not provide our own proof. It follows that the η-polynomial obstructs many strongly invertible slice knots from being equivariantly slice. But beyond this η does not give information on the equivariant 4-genus.

Theorem 1.1 gives an alternative proof of the analogous smooth result, that the smooth equivariant 4-genus can be arbitrarily large for slice knots, due to Dai–Mallick–Stoffregen [15], and proven using knot Floer homology. Our methods do not recover their specific examples, but tend to require significantly easier computations, as evidenced by the fact that Theorem 1.1 applies to a large class of strongly invertible knots. In [15], they also consider the *isotopy-equivariant 4-genus*, where one relaxes the condition that $\widehat{\tau}(\Sigma) = \Sigma$ to instead to require that $\widehat{\tau}(\Sigma)$ is ambiently isotopic to Σ while keeping the boundary pointwise fixed. Our lower bounds extend to this setting with identical proofs.

Recent work of Boyle–Issa [2] used Donaldson's theorem to give a lower bound on the smooth equivariant 4-genus, but this bound is only capable of establishing a gap of one between the usual smooth 4-genus and the smooth equivariant 4-genus, and therefore cannot prove Theorem 1.1 even in the smooth category. There has been further significant recent interest in equivariant concordance of strongly invertible knots, including by Dai–Hedden–Mallick [14], Alfieri–Boyle [1], and Di Prisa [16], but none of this work gives lower bounds on the equivariant 4-genus.

Theorem 1.1 is a consequence of our main obstruction theorem, which reads as follows. The *generating rank* of a $\mathbb{Q}[t^{\pm 1}]$-module Q, denoted $\operatorname{g-rk} Q$, is by definition the number of elements in a generating set of minimal cardinality. The (rational) *Alexander module* of an oriented knot K is the first homology $\mathcal{A}(K) := H_1(E_K; \mathbb{Q}[t^{\pm 1}])$ of the infinite cyclic cover of the knot exterior E_K. This admits a nonsingular, Hermitian, sesquilinear *Blanchfield pairing* [3] $\mathcal{B}\ell_K : \mathcal{A}(K) \times \mathcal{A}(K) \to \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$, whose definition we will recall in detail in Section 2.

Theorem 1.2. *Let (K, τ) be a strongly invertible knot. Let k be the maximal generating rank of any submodule P of $\mathcal{A}(K)$ satisfying $\mathcal{B}\ell_K(x, y) = 0 = \mathcal{B}\ell_K(x, \tau_*(y))$ for all $x, y \in P$. Then*

$$\widetilde{g}_4(K, \tau) \geqslant \frac{\operatorname{g-rk} \mathcal{A}(K) - 2k}{4}.$$

To apply this lower bound to a given knot, one only needs to make a relatively straightforward computation of the Blanchfield pairing, which can be done in terms of a Seifert matrix using [21, 27]. To prove Theorem 1.1, we compute that $k = 0$ and $\operatorname{g-rk} \mathcal{A}(K) = n$ when $K = \#^n J$ and J is a genus one algebraically slice knot with Δ_J nontrivial.

1.2 | An equivariant algebraic knot concordance group

A *direction* for a strongly invertible knot (K, τ) consists of a choice of orientation of the great circle γ and a choice of connected component of $\gamma \setminus K$. The set of directed, strongly invertible knots admits a well-defined connected sum, with respect to which it forms a group if we quotient by the relation of equivariant concordance. Here two strongly invertible knots (K_1, τ_1) and (K_2, τ_2) are *equivariantly concordant* if there is a locally flat concordance $A \subseteq S^3 \times I$ between K_1 and K_2 together with an extension of τ_1 and τ_2 to an involution $\widehat{\tau} : S^3 \times I \to S^3 \times I$ with $\widehat{\tau}(A) = A$, and such that the directions are preserved. We give more details in Subsection 2.3.

Taking the Blanchfield form $\mathcal{B}\ell_K^{\mathbb{Z}}$ of a knot K gives rise to a homomorphism from the knot concordance group to the algebraic knot concordance group, $\mathcal{B}\ell : \mathcal{C} \to \mathcal{AC}$. The latter is the Witt group of abstract Blanchfield forms, which is isomorphic to the possibly more familiar Witt group of Seifert forms, see [18, 27, 28, 46], as well as [39] for a purely algebraic proof. The analogous homomorphism for odd high-dimensional knots S^{2k-1} in S^{2k+1} is an isomorphism for $k \geqslant 2$ [13, 28]. For $k = 1$ the algebraic concordance group has been the framework for deeper investigation of $\mathcal{C}$, see, for example, [6–9, 11, 17, 23, 26, 30–32, 35, 36].

The Blanchfield form interpretation of the algebraic concordance group lends itself to generalization to the equivariant setting. Let $\mathcal{C}^{SI}$ denote the equivariant concordance group of strongly invertible knots. We define an *equivariant algebraic concordance group* $\mathcal{AC}^{SI}$ by considering a Witt group of abstract Blanchfield forms $(H, \mathcal{B}\ell)$ endowed with an anti-isometry $\tau : H \to H$, and requiring metabolizers to be τ-invariant. We give the detailed definition of $\mathcal{AC}^{SI}$ and a homomorphism $\Psi : \mathcal{C}^{SI} \to \mathcal{AC}^{SI}$ in Section 4.2. This fits into a commutative diagram, where the vertical maps forget the inversion and the horizontal maps pass from geometry to algebra.

$$\begin{tikzcd} \mathcal{C}^{SI} \arrow[r, "\Psi"] \arrow[d, "F"] & \mathcal{AC}^{SI} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{AC}. \end{tikzcd}$$

The bottom homomorphism is surjective [28] and has kernel of infinite rank [26].

Theorem 1.3. *There is a subgroup of* $\ker(F : \mathcal{C}^{SI} \to \mathcal{C})$ *of infinite rank, and whose image in* $\mathcal{AC}^{SI}$ *is infinite rank.*

Remark 1.4. What else do we know about the maps in the square above?

(1) It follows from [33] that F is not surjective. If it were, every knot would be concordant to a reversible knot, but a knot that is concordant to a reversible knot is concordant to its own reverse, and Livingston found knots not concordance to their own reverses.
(2) We do not know whether Ψ is surjective, nor whether the forgetful map $\mathcal{AC}^{SI} \to \mathcal{AC}$ is surjective. However, some evidence toward the surjectivity of Ψ was given by Sakai, who showed [40] that every Alexander polynomial is realized by a strongly invertible knot.
(3) Sakuma's η-invariant [41] was already an effective way to obstruct knots from being equivariantly slice, and Sakuma used it to show that $\ker F$ is nontrivial, for example by showing that the Stevedore knot is not equivariantly slice. Moreover, he showed that for K the untwisted Whitehead double of the trefoil and figure eight knots, $\eta(K) \neq 0$, and therefore that $\ker F \cap \ker \Psi$ is nontrivial. Note that these examples are isotopy-equivariantly (topologically) slice by [12], as they have Alexander polynomial one.
(4) All of $\mathcal{AC}^{SI}$, $\mathcal{AC}$, and $\mathcal{C}$ are abelian, whereas $\mathcal{C}^{SI}$ is not [16]. So, the nontrivial commutators found by Di Prisa also lie in $\ker F \cap \ker \Psi$.

Levine and Stoltzfus [28, 45] algebraically computed that $\mathcal{AC} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty$. We aim to analyze the isomorphism type of $\mathcal{AC}^{SI}$ in future work.

In our proof of Theorem 1.3, we use genus one knots to exhibit the claimed subgroup of $\ker F$. The proof also shows the following result, which we think worth emphasizing.

Corollary 1.5. *Let K be a genus one strongly invertible knot with nontrivial Alexander polynomial. Then K is not equivariantly slice.*

This is also formally a consequence of Theorem 1.1, although we prove the corollary before we start the proof of Theorem 1.1.

Organization of the paper

In Section 2, we recall the Alexander module and the Blanchfield form of a knot, and we show that a strong involution τ induces an anti-isometry of the Blanchfield form. We also recall the definition of equivariant connected sum, and consider its effect on these data. In Section 3, we perform some detailed computations of Blanchfield forms for some key examples. In Section 4.2, we show that the Blanchfield form of an equivariantly slice strongly invertible knot has an equivariant metabolizer. We use this observation to motivate the definition of an equivariant algebraic concordance group $\mathcal{AC}^{SI}$, and we then prove Theorem 1.3 and Corollary 4.2. We also give an infinite family of amphichiral knots that are infinite order in $\mathcal{AC}^{SI}$. Finally, Section 5 contains the proof of the lower bound in Theorem 1.2, and then by combining this theorem with the computations in Section 3, we deduce Theorem 1.1.

2 | ALEXANDER MODULES, BLANCHFIELD FORMS, EQUIVARIANT CONNECTED SUM, AND EQUIVARIANT CONCORDANCE

In this section, we recall the equivariant connected sum of strongly invertible knots, and we deduce algebraic conclusions, on the level of Alexander modules and Blanchfield pairings, from the existence of a strongly invertible slice disc. Throughout this section and the remainder of the article, we write $\Lambda := \mathbb{Q}[t^{\pm 1}]$.

Definition 2.1. Given a finitely generated Λ-module U, we define the following notions.

(1) By the fundamental theorem on finitely generated modules over a PID, there exists some $n, m \geqslant 0$ and $p_1(t), \dots, p_m(t) \in \Lambda$ such that

$$U \cong \Lambda^n \oplus \bigoplus_{i=1}^{m} \Lambda/p_i(t).$$

When $n = 0$, the order of U is defined to be $\prod_{i=1}^{m} p_i(t)$; otherwise, the order is defined to be 0. In both cases, we denote the order of U by $|U|$; this is an element of Λ well-defined up to multiplication by units of Λ.

(2) The Λ-module $\overline{U}$ setwise agrees with U and has Λ-action defined by $p(t) \cdot_{\overline{U}} u = \overline{p(t)} \cdot_U u$ for all $p(t) \in \Lambda$ and $u \in U$, where $\overline{\cdot}$ is the $\mathbb{Q}$-linear involution on Λ sending t^k to t^{-k} for all $k \in \mathbb{Z}$.

2.1 | The involution induced on the Alexander module

Let K be an oriented knot in S^3 and let $E_K := S^3 \setminus \nu K$ denote the exterior of K. Let $M_K := S^3_0(K)$ denote the result of 0-framed surgery on S^3 along K. Let μ_K be an oriented meridian for K and let λ_K be a 0-framed oriented longitude. Requiring that μ_K maps to $t \in C_\infty \cong \langle t \rangle$ determines surjections factoring through the Hurewicz maps:

$$\pi_1(E_K) \to H_1(E_K; \mathbb{Z}) \cong \mathbb{Z} \cong \langle t \rangle,$$
$$\pi_1(M_K) \to H_1(M_K; \mathbb{Z}) \cong \mathbb{Z} \cong \langle t \rangle.$$

These in turn determine coefficient systems for twisted homology:

$$H_i(E_K; \Lambda) := H_i(\Lambda \otimes_{\mathbb{Z}[\pi_1(E_K)]} C_*(\widetilde{E}_K)),$$
$$H_i(M_K; \Lambda) := H_i(\Lambda \otimes_{\mathbb{Z}[\pi_1(M_K)]} C_*(\widetilde{M}_K)).$$

One makes analogous constructions for $\mathbb{Z}[t^{\pm 1}]$ in place of $\Lambda = \mathbb{Q}[t^{\pm 1}]$.

Moreover, given a 4-manifold Z with $\partial Z = M_K$ such that the inclusion induced map $i_* : H_1(M_K;\mathbb{Z}) \to H_1(Z;\mathbb{Z})$ is an isomorphism, we will frequently consider $H_1(Z;\Lambda) := H_i(\Lambda \otimes_{\mathbb{Z}[\pi_1(Z)]} C_*(\widetilde{Z}))$ and a similarly defined $H_1(Z, M_K;\Lambda)$, where the coefficient system on Z is given by composing i_*^{-1} with our standard map $H_1(M_K;\mathbb{Z}) \xrightarrow{\cong} \langle t \rangle$.

Definition 2.2. The (rational) *Alexander module* of K is the Λ-module $\mathcal{A}(K) := H_1(E_K;\Lambda)$. The order of this Λ-module is exactly the Alexander polynomial of K. Observe that the 0-framed longitude of K lifts to the infinite cyclic cover of E_K and therefore defines a class in $H_1(E_K;\Lambda)$, and this homology class is trivial because any Seifert surface for K exhibits the longitude as a double commutator in $\pi_1(E_K)$. Therefore, the inclusion-induced map $H_1(E_K;\Lambda) \to H_1(M_K;\Lambda)$ is an isomorphism, and we could have equally well-defined the Alexander module of K to be $H_1(M_K;\Lambda)$.

The *integral Alexander module* of K is $\mathcal{A}^{\mathbb{Z}}(K) := H_1(E_K;\mathbb{Z}[t^{\pm 1}])$, or equivalently $H_1(M_K;\mathbb{Z}[t^{\pm 1}])$, for the same reason as above.

A strong involution determines some additional algebraic structure on the Alexander module, of the following type.

Definition 2.3. Let U be a Λ-module. A Λ-module isomorphism $f : U \to \overline{U}$ is called an *anti-automorphism*. That is, f is a $\mathbb{Q}$-linear bijection such that

$$f(t^k \cdot_U x) = t^k \cdot_{\overline{U}} f(x) = t^{-k} \cdot_U f(x)$$

for all $k \in \mathbb{Z}$ and $x \in U$. An analogous definition holds for $\mathbb{Z}[t^{\pm 1}]$-modules in place of Λ-modules.

Let K be a strongly invertible knot with involution τ. Restricting τ to E_K gives a homeomorphism $\tau_K : E_K \to E_{K^r}$ that sends μ_K to μ_{K^r} and λ_K to λ_{K^r}. For every knot J, the identity map on S^3 restricts to a function $\rho_J : E_J \to E_{J^r}$ that sends μ_J to μ_J^{-1} and λ_J to λ_J^{-1}. Therefore, we can consider the composition

$$\rho_K^{-1} \circ \tau : E_K \to E_K,$$

an orientation preserving homeomorphism of E_K that squares to the identity map and sends μ_K to μ_K^{-1}, and λ_K to λ_K^{-1}.

Further, the map $\rho_K^{-1} \circ \tau$ induces an anti-automorphism of $\mathcal{A}(K)$, as in Definition 2.3, which by a mild abuse of notation we refer to as $\tau_* : \mathcal{A}(K) \to \mathcal{A}(K)$.

Definition 2.4. Given a strongly invertible knot (K, τ), let

$$\tau_* := (\rho_K^{-1} \circ \tau)_* : \mathcal{A}(K) \to \mathcal{A}(K)$$

be defined as above. We call the anti-automorphism τ_* the *inversion-induced map on the Alexander module*. There is an analogous anti-automorphism $\tau_* : \mathcal{A}^{\mathbb{Z}}(K) \to \mathcal{A}^{\mathbb{Z}}(K)$.

Example 2.5. We give some examples of inversion-induced maps.

(i) Let $K = 6_1$ be the Stevedore knot illustrated in Figure 2. Standard arguments using Seifert matrices or Fox calculus (see also Proposition 3.1) imply that $\mathcal{A}(K) \cong \Lambda/(2t - 5 + 2t^{-1})$, with

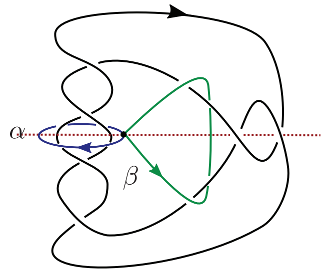

FIGURE 2 The Stevedore knot $K = 6_1$, illustrated with its axis of inversion and curves α and β, each of whose lift generates $\mathcal{A}(K)$.

generator given by the class represented by $\widetilde{\alpha}$, the lift of the curve α illustrated in Figure 2 to the infinite cyclic cover of the knot exterior. As the involution τ sends α to itself in an orientation reversing way while fixing our chosen basepoint for E_K, we obtain that $\tau_*([\widetilde{\alpha}]) = -[\widetilde{\alpha}]$. As τ_* is an anti-automorphism, we see that $\tau_*(p(t)[\widetilde{\alpha}]) = -p(t^{-1})[\widetilde{\alpha}]$ for all $p(t) \in \Lambda$.

Note that this description of τ_* depends on the generator we pick for $\mathcal{A}(K)$. For example, if we had chosen the lift of β as our preferred generator for $\mathcal{A}(K)$, we would have computed that $\tau * ([\widetilde{\beta}]) = t^{-1}[\widetilde{\beta}]$, as τ sends β to $\mu^{-1}\beta\mu$ for a certain meridian μ of K. We therefore could have alternately said that $\mathcal{A}(K) \cong \Lambda/(2t - 5 + 2t^{-1})$, with τ_* the unique anti-automorphism sending the generator $[\widetilde{\beta}]$ to $t^{-1}[\widetilde{\beta}]$.

The action of τ_* on $[\widetilde{\beta}]$ can also be computed using the fact, verifiable via a Seifert surface for K, that $[\widetilde{\beta}] = (2t-2)[\widetilde{\alpha}]$ in $\mathcal{A}(K)$. Then we can observe

$$\tau_*([\widetilde{\beta}]) = (2t^{-1} - 2)\tau_*([\widetilde{\alpha}]) = (2 - 2t^{-1})[\widetilde{\alpha}] = t^{-1}(2t-2)[\widetilde{\alpha}] = t^{-1}[\widetilde{\beta}],$$

as asserted.

(ii) Let $K = J\#J^r$ and let τ be the involution which switches the two factors. Then $\mathcal{A}(K) \cong \mathcal{A}(J) \oplus \mathcal{A}(J^r)$ and $\tau_*(x, y) = (y, x)$. Observe that for any $(x, y) \in A(J) \oplus \mathcal{A}(J^r)$ one has

$$\tau_*(t_{J\#J^r} \cdot (x, y)) = \tau_*(t_J \cdot x, t_{J^r} \cdot y) = (t_{J^r} \cdot y, t_J \cdot x) = (t_J^{-1} \cdot y, t_{J^r}^{-1} \cdot x) = t_{J\#J^r}^{-1} \cdot \tau_*(x, y),$$

so we can directly verify that τ_* is an anti-automorphism.

2.2 | The Blanchfield pairing

As above let M_K be the result of zero-framed surgery on S^3 along a knot K. We now briefly outline the definition of the Blanchfield pairing, referring the reader to [34] for more details. We consider the sequence of isomorphisms of Λ-modules

$$\Theta : H_1(M_K; \Lambda) \xrightarrow{PD^{-1}} H^2(M_K; \Lambda) \xrightarrow{B^{-1}} H^1(M_K; \mathbb{Q}(t)/\Lambda) \xrightarrow{\kappa} \overline{\mathrm{Hom}_\Lambda}(H_1(M_K; \Lambda), \mathbb{Q}(t)/\Lambda).$$

These maps are given respectively by the inverse of Poincaré duality, the inverse of the Bockstein isomorphism corresponding to the short exact sequence of coefficients $0 \to \Lambda \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\Lambda \to 0$, and the evaluation map. The Bockstein map B is an isomorphism because

$H_i(M_K;\mathbb{Q}(t)) = 0$ for $i = 1, 2$, as $H_i(M_K;\Lambda)$ is Λ-torsion [29]. The evaluation map κ is an isomorphism by the universal coefficient theorem, because $\mathbb{Q}(t)/\Lambda$ is an injective Λ-module and so $\mathrm{Ext}^1_\Lambda(H_0(M_K;\Lambda), \mathbb{Q}(t)/\Lambda) = 0$.

Definition 2.6. The Blanchfield pairing $\mathcal{B}\ell_K : H_1(M_K;\Lambda) \times H_1(M_K;\Lambda) \to \mathbb{Q}(t)/\Lambda$ is given by $\mathcal{B}\ell_K(x, y) := \Theta(y)(x)$.

The Blanchfield pairing, originally defined in [3], is sesquilinear, Hermitian, and nonsingular; see, for example, [37] for more details. Here sesquilinear means that $\mathcal{B}\ell_K(px, qy) = p\mathcal{B}\ell_K(x, y)\overline{q}$ and Hermitian means that $\mathcal{B}\ell_K(y, x) = \overline{\mathcal{B}\ell_K(x, y)}$, for all $x, y \in H_1(M_K;\Lambda)$ and for all $p, q \in \Lambda$. The analogous definition applies with $\mathbb{Z}[t^{\pm 1}]$ replacing Λ, giving rise to the integral Blanchfield form $\mathcal{B}\ell^{\mathbb{Z}}_K$. It is more work to show that the evaluation map is an isomorphism in this case, but it still holds, see, for example, [29].

Given a Λ-module U and a sesquilinear, Hermitian pairing $B : U \times U \to \mathbb{Q}(t)/\Lambda$, there is an involuted pairing $\overline{B} : \overline{U} \times \overline{U} \to \mathbb{Q}(t)/\Lambda$ given by $\overline{B}(x, y) = \overline{B(x, y)}$. The pairing $\overline{B}$ is sesquilinear in the sense that $\overline{B}(px, qy) = \overline{p}\overline{B}(x, y)q$.

Definition 2.7. Let U be a Λ-module and let $B : U \times U \to \mathbb{Q}(t)/\Lambda$ be a sesquilinear, Hermitian pairing. An anti-automorphism $f : U \to \overline{U}$ is called an *anti-isometry* of (U, B) if

$$B(x, y) = \overline{B}(f(x), f(y)) = \overline{B(f(x), f(y))}$$

for all $x, y \in U$. That is, an anti-isometry induces an isometry between $B : U \times U \to \mathbb{Q}(t)/\Lambda$ and $\overline{B} : \overline{U} \times \overline{U} \to \mathbb{Q}(t)/\Lambda$. An analogous definition holds for $\mathbb{Z}[t^{\pm 1}]$-modules and a pairing with values in $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$.

We saw in the previous section that $\tau_* : H_1(M_K;\Lambda) \to H_1(M_K;\Lambda)$ is an anti-automorphism, and now prove the following.

Proposition 2.8. *Let (K, τ) be a strongly invertible knot. The inversion-induced map on the Alexander module, τ_*, induces an anti-isometry of the Blanchfield pairing $\mathcal{B}\ell_K$. The same holds for the integral Alexander module and $\mathcal{B}\ell^{\mathbb{Z}}_K$.*

Proof. The homeomorphism $\tau : E_K \to E_{K^r}$ induces an isometry of Blanchfield pairings $\mathcal{B}\ell_K \cong \mathcal{B}\ell_{K^r}$. The map $\rho_K : E_{K^r} \to E_K$ identifies $\mathcal{B}\ell_{K^r}$ with $\overline{\mathcal{B}\ell_K}$. Therefore, $\rho_K^{-1} \circ \tau$, the map which induces $\tau_* : \mathcal{A}(K) \to \mathcal{A}(K)$, induces an isometry of $\mathcal{B}\ell_K$ with $\overline{\mathcal{B}\ell_K}$, or in other words an anti-isometry of $\mathcal{B}\ell_K$, as required. □

2.3 | Equivariant connected sum of knots

We recall the definition of the connected sum of two directed strongly invertible knots, following Sakuma [41]. A *direction* on a strongly invertible knot (K, τ) is a choice of orientation of the great circle γ, and a choice of one of the two connected component of $\gamma \setminus K$. A strongly invertible knot together with a choice of direction is called *directed*. These extra data enable us to remove the indeterminacy in the definition of connected sum.

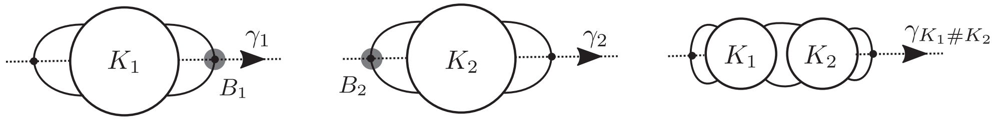

FIGURE 3 Two directed strongly invertible knots (left and middle) and their connected sum (right). For each of the three knots, the preferred component of the axis of symmetry passes through the point at infinity.

Definition 2.9 (Equivariant connected sum). Let (K_1, τ_1) and (K_2, τ_2) be directed strongly invertible knots, so $\tau_i : (S^3, K_i) \to (S^3, K_i)$ is rotation by π about the axis γ_i. As illustrated in Figure 3, for $i = 1, 2$ let B_i be a small neighborhood of one of the intersection points of K_i with γ_i. For K_1, use the intersection point that lies at the *start* of the chosen connected component of $\gamma_1 \setminus K_1$. For K_2, use the intersection point that lies at the *end* of the chosen connected component of $\gamma_2 \setminus K_2$. Arrange that $\overline{B}_i \cap K_i$ is an unknotted arc and that τ_i restricts to a homeomorphism of pairs $(\overline{B}_i, \overline{B}_i \cap K_i)$. Let

$$f : \partial\left(\overline{B}_1, \overline{B}_1 \cap K_1, \overline{B}_1 \cap \gamma_1\right) \to \partial\left(\overline{B}_2, \overline{B}_2 \cap K_2, \overline{B}_2 \cap \gamma_2\right)$$

be a homeomorphism of triples such that:

(i) f is an orientation-reversing homeomorphism of S^3;
(ii) $f \circ \tau_1 \circ f^{-1} = \tau_2$;
(iii) the point of $\overline{B}_1 \cap \gamma_1$ at which the orientation of γ_1 points into B_1 is identified with the point of $\overline{B}_2 \cap \gamma_2$ at which the orientation of γ_2 points out of B_2.

Then there is a homeomorphism of triples

$$\begin{aligned}&\left((S^3, K_1, \gamma_1) \setminus (B_1, B_1 \cap K_1, B_1 \cap \gamma_1)\right) \cup_f \left((S^3, K_2, \gamma_2) \setminus (B_2, B_2 \cap K_2, B_2 \cap \gamma_2)\right)\\&\cong \left(S^3, K_1\#K_2, \gamma_1\#\gamma_2\right),\end{aligned}$$

which defines the equivariant connected sum $K_1\#K_2$. This comes with a strong involution τ obtained from gluing τ_1 and τ_2, with fixed set $\gamma_1\#\gamma_2$ and such that $\tau(K_1\#K_2) = K_1\#K_2$.

To define the direction on the connected sum, we take the orientation on $\gamma_1\#\gamma_2$ induced by the orientations of γ_1 and γ_2, and we take the connected component of $\gamma_1\#\gamma_2$ which contains the original preferred components of γ_1 and γ_2 (minus $B_1 \cap \gamma_1$ and $B_2 \cap \gamma_2$, respectively).

We call (K, τ) the *equivariant connected sum* of (K_1, τ_1) and (K_2, τ_2). Sakuma [41, section 1] proved that the equivariant isotopy class of $K_1\#K_2$ does not depend on the choice of f satisfying the above conditions.

Remark 2.10. In particular, note that we did not fix an orientation on K_1 nor on K_2. As strongly invertible knots are reversible, $K_1\#K_2$ is isotopic to $K_1\#K_2^r$. As indicated in [41, figure 1.2], the two knots are moreover equivariantly isotopic, and so it is not necessary to choose orientations on the K_i.

Definition 2.11 (Equivariant concordance). Let (K_0, τ_0) and (K_1, τ_1) be directed strongly invertible knots with axes γ_0 and γ_1, respectively.

(a) Suppose that there is a concordance between K_0 and K_1, that is, there is a locally flat embedding $c: S^1 \times [0,1] \to S^3 \times [0,1]$ with $c(S^1 \times \{i\}) = K_i$ for $i = 0, 1$. The image $C := c(S^1 \times [0,1]$ is a proper submanifold of $S^3 \times [0,1]$.
(b) Suppose also that there is an involution of $(S^3 \times [0,1], C)$ extending τ_0 and τ_1, that is an order two locally linear homeomorphism $\widehat{\tau}: S^3 \times [0,1] \to S^3 \times [0,1]$ such that $\widehat{\tau}|_{S^3 \times \{i\}} = \tau_i : S^3 \times \{i\} \to S^3 \times \{i\}$ for $i = 0, 1$, and such that $\widehat{\tau}(C) = C$.
(c) Let A be the set of fixed points of $\widehat{\tau}$. By Remark 2.12, A is a locally flat concordance between γ_0 and γ_1. Suppose that the chosen connected components of $\gamma_0 \setminus K_0$ and $\gamma_1 \setminus K_1$ lie in the same connected component of $A \setminus C$, and that the orientations of γ_0 and γ_1 induce opposite orientations on A.

Then we say that (K_0, τ_0) and (K_1, τ_1) are *directed equivariantly concordant.*

Remark 2.12. We explain why the fixed point set A of $\mathbb{Z}/2$ acting on $S^3 \times [0,1]$ via the involution $\widehat{\tau}$ is an annulus. As $\widehat{\tau}$ is locally linear, the fixed set A is a submanifold of locally constant dimension. Let $(W, f) = (S^3 \times I, \widehat{\tau}) \cup (D^4, \sigma)$, where σ is the standard extension of τ to D^4.

Smith's theorem on finite group actions [43, 44] (see also [4, section III, Theorem 5.2] for a modern treatment) implies that the fixed set of (W, f) is a $\mathbb{Z}/2$-homology ball, and in particular is connected. As the fixed set restricts to γ_0 in the boundary S^3, it is a connected surface with boundary γ_0, and as it is a $\mathbb{Z}/2$ homology ball it must be homeomorphic to D^2. Remove (D^4, σ), and note that the fixed set of σ is also a disc, with boundary γ_1, to see that the fixed set A of $\widehat{\tau}$ is indeed an annulus with $\partial A = \gamma_0 \cup -\gamma_1$.

Now that we have a well-defined notion of equivariant connected sum and equivariant concordance, we can define the equivariant concordance group. See also [41] and, for example, [2, section 2] and [15, section 2.1].

Definition 2.13 (Equivariant concordance group). The set of directed equivariant concordance classes of directed strongly invertible knots forms a group under equivariant connected sum, as observed by Sakuma [41]. The identity element is the directed equivariant concordance class of the unknot, and the inverse of the directed strongly invertible knot (K, τ) is the knot obtained by reversing the orientations of S^3, with the direction given by reversing the orientation of γ and keeping the same preferred component. We denote the *equivariant concordance group* by $\mathcal{C}^{SI}$.

As explained in the upcoming proposition, the choice of directions do not affect the Alexander module nor the Blanchfield form of an equivariant connected sum, and therefore while they are necessary in order to define $\mathcal{C}^{SI}$, we will not need to focus on them in the rest of the paper.

Proposition 2.14. *Let (K_1, τ_1) and (K_2, τ_2) be strongly invertible knots. For any choice of directions on K_1 and K_2, the Alexander module and Blanchfield pairing of the equivariant sum $(K_1, \tau_1)\#(K_2, \tau_2)$ is the direct sum $(\mathcal{A}(K_1) \oplus \mathcal{A}(K_2), \mathcal{B\ell}_{K_1} \oplus \mathcal{B\ell}_{K_2})$, and with respect to this identification the induced involution is $(\tau_{K_1\#K_2})_* = (\tau_{K_1})_* \oplus (\tau_{K_2})_*$. The same holds for the integral versions $\mathcal{A}^{\mathbb{Z}}(K)$ and $\mathcal{B\ell}^{\mathbb{Z}}_K$.*

Proof. The exterior of $K_1\#K_2$ can be obtained by gluing the exteriors of K_1 and K_2 along a thickened oriented meridian for each (or, to use the perspective of Definition 2.9, gluing the exterior of the knotted arc for K_1 to that of the knotted arc for K_2). A thickened meridian μ has $H_i(\mu; \Lambda) = 0$

for $i \geqslant 1$, and $H_0(\mu;\Lambda) \cong \mathbb{Q}$ so the Alexander modules add by a Mayer–Vietoris argument. As $\tau_{K_1\#K_2}$ is defined by gluing τ_1 on E_{K_1} and τ_2 on E_{K_2}, it follows that it induces $(\tau_{K_1})_* \oplus (\tau_{K_2})_*$ on $\mathcal{A}(K_1) \oplus \mathcal{A}(K_2) \cong \mathcal{A}(K_1\#K_2)$. It is well-known that the Blanchfield pairing of a connected sum is the direct sum as claimed. For example, one can see this using the fact that the Seifert forms add in this way, and that the Blanchfield pairing can be computed using the Seifert pairing [27] (see also [21]). Alternatively, one can apply [19]. □

3 | COMPUTATIONS OF THE BLANCHFIELD PAIRING

In this section, we explicitly compute the Blanchfield pairing for specific families of strongly invertible knots, in particular for every genus one algebraically slice knot. We will make use of these computations in the proofs of our main results. To avoid interrupting the arguments later, and to be able to appeal to these computations in Sections 4.2 and 5, we collect these computations first here.

The following proposition can be deduced by combining [21, Theorems 1.3 and 1.4], and by passing from $\mathbb{Z}[t^{\pm 1}]$ to $\mathbb{Q}[t^{\pm 1}]$ coefficients.

Proposition 3.1 (Friedl–Powell). *Let F be a Seifert surface for a knot K with a collection of simple closed curves $\alpha_1, \dots, \alpha_{2g}$ on F that form a basis for $H_1(F;\mathbb{Z})$, and let A be the corresponding Seifert matrix. Let $\beta_1, \dots, \beta_{2g}$ be a dual basis for $H_1(S^3 \setminus \nu(F);\mathbb{Z})$, that is, a basis such that $\operatorname{lk}(\alpha_i, \beta_j) = \delta_{i,j}$. Using the standard decomposition of $(S^3 \setminus \nu(K))_\infty = \cup_{j=-\infty}^{\infty} (S^3 \setminus \nu(F))_j$, let the homology class of the unique lift of β_i to $(S^3 \setminus \nu(F))_0$ be denoted by b_i. Then the map $p:\ (\mathbb{Q}[t^{\pm 1}])^{2g} \to \mathcal{A}(K)$ given by $p(x_1, \dots, x_{2g}) = \sum_{i=1}^{2g} x_i b_i$ is a surjective map with kernel $(tA - A^T)(\mathbb{Q}[t^{\pm 1}])^{2g}$. Moreover, for $x, y \in \mathbb{Q}[t^{\pm 1}]^{2g}$ the rational Blanchfield pairing is given by*

$$\mathcal{B}\ell(p(x), p(y)) = (t-1)x^T(A - tA^T)^{-1}\overline{y},$$

where $\bar{\cdot}$ is the component-wise extension of the $\mathbb{Q}$-linear involution on $\mathbb{Q}[t^{\pm 1}]$ sending t^i to t^{-i}.

We will use the following elementary fact to verify that certain elements of $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ are nonzero.

Lemma 3.2. *Let $p, q \in \mathbb{Q}[t^{\pm 1}]$ be coprime. Then $\frac{a}{p} + \frac{b}{q} \in \mathbb{Q}[t^{\pm 1}] \subseteq \mathbb{Q}(t)$ if and only if both $\frac{a}{p}$ and $\frac{b}{q}$ belong to $\mathbb{Q}[t^{\pm 1}]$.*

Proof. The if direction is trivial. If $\frac{a}{p} + \frac{b}{q} \in \mathbb{Q}[t^{\pm 1}]$ then $\frac{aq+bp}{pq} \in \mathbb{Q}[t^{\pm 1}]$, so $p \mid aq + bp$, which implies $p \mid aq$. As $\mathbb{Q}[t^{\pm 1}]$ is a PID and hence a UFD, the fact that p and q are coprime implies that $p \mid a$. Therefore, $\frac{a}{p} \in \mathbb{Q}[t^{\pm 1}]$. By symmetry this suffices. □

Example 3.3 (The knot 9_{46}). The knot 9_{46} is shown in Figure 4. Denote the depicted generators for $H_1(F)$ by α_1 and α_2 and the depicted dual generating set for $H_1(S^3 \setminus F)$ by β_1 and β_2. The Seifert matrix for F with respect to this basis is given by $A = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$. By Proposition 3.1, we therefore have that

$$\mathcal{A}(K) \cong \mathbb{Q}[t^{\pm 1}]/(t-2) \oplus \mathbb{Q}[t^{\pm 1}]/(2t-1),$$

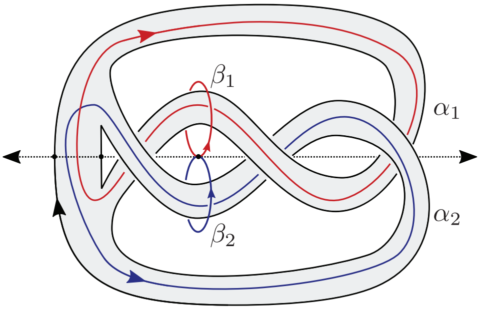

FIGURE 4 The knot $K = 9_{46}$ with an axis of inversion and a Seifert surface.

where the first summand is generated by b_1 and the second summand by b_2. Observe that as τ fixes the illustrated basepoint for E_K and $\tau(\beta_1) = \beta_2$ and $\tau(\beta_2) = \beta_1$, we have that $\tau_* : \mathcal{A}(K) \to \mathcal{A}(K)$ sends b_1 to b_2 and b_2 to b_1. More precisely, for any $p_1(t), p_2(t) \in \mathbb{Q}[t^{\pm 1}]$, we have that

$$\tau_*(p_1(t)b_1 + p_2(t)b_2) = p_2(t^{-1})b_1 + p_1(t^{-1})b_2.$$

Note that every element x of $\mathcal{A}(K)$ can be written as $x = c_1 b_1 + c_2 b_2$ for some $c_1, c_2 \in \mathbb{Q}$ because as abelian groups $\mathbb{Q}[t^{\pm 1}]/(t-2) \cong \mathbb{Q} \cong \mathbb{Q}[t^{\pm 1}]/(2t-1)$. We now compute $\mathcal{B}\ell(x, \tau_*(x))$ using Proposition 3.1:

$$\begin{aligned}
\mathcal{B}\ell(x, \tau_*(x)) &= \mathcal{B}\ell(c_1 b_1 + c_2 b_2, c_2 b_1 + c_1 b_2) \\
&= (t-1)\begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} 0 & 2-t \\ 1-2t & 0 \end{pmatrix}^{-1} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} \\
&= \frac{-(t-1)}{(2t-1)(t-2)} \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} 0 & t-2 \\ 2t-1 & 0 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} \\
&= -(t-1)\left(\frac{(c_1)^2}{2t-1} + \frac{(c_2)^2}{t-2} \right).
\end{aligned}$$

Applying Lemma 3.2, we deduce that $\mathcal{B}\ell(x, \tau_*(x)) = 0$ if and only if $c_1 = c_2 = 0$, that is, $x = 0$.

We can generalize Example 3.3 to the following result.

Proposition 3.4. *Let K be a genus one algebraically slice knot with strong inversion τ and nontrivial Alexander polynomial. For each $n \in \mathbb{N}$, let $(K_n, \tau_n) := \#^n(K, \tau)$. For every nonzero $x \in \mathcal{A}(K_n)$, we have that $\mathcal{B}\ell(x, (\tau_n)_*(x)) \neq 0$.*

Proof. We begin by constraining the action of τ_* on $\mathcal{A}(K)$. As K is algebraically slice and genus one, it has some Seifert surface F and a basis for $H_1(F)$ with respect to which its Seifert matrix is $A = \left(\begin{smallmatrix} 0 & m+1 \\ m & \ell \end{smallmatrix}\right)$ for some $m, \ell \in \mathbb{Z}$. By further change of basis of $H_1(F)$, we can assume that $\ell \neq 0$.

Using $tA - A^T$ to present $\mathcal{A}(K)$ as in Proposition 3.1, we see that $\mathcal{A}(K)$ is generated as a $\mathbb{Q}[t^{\pm 1}]$ module by b_1 and b_2, subject to the relations

$$0 = (mt - (m+1))b_2 \text{ and}$$
$$0 = ((m+1)t - m)b_1 + \ell(t-1)b_2.$$

Adding $-\frac{\ell}{m}$ times the first equation to the second and solving for b_2 gives us that

$$b_2 = \frac{m}{\ell}((m+1)t - m))b_1,$$

and hence that $\mathcal{A}(K)$ is cyclic with generator b_1. As the order of $\mathcal{A}(K)$ is exactly the Alexander polynomial, which is given by $\det(tA - A^T)$, we obtain that

$$\mathcal{A}(K) \cong \mathbb{Q}[t^{\pm 1}]/(mt - (m+1))((m+1)t - m),$$

with b_1 as a generator. As K has nontrivial Alexander polynomial we know that $m, (m+1) \neq 0$. Let $y_1 := ((m+1)t - m)b_1$ and $y_2 := (mt - (m+1))b_1$. One can now compute that $ty_1 = \frac{m+1}{m}y_1$ and $ty_2 = \frac{m}{m+1}y_2$.

Now recall that $\tau_* : \mathcal{A}(K) \to \mathcal{A}(K)$ is a $\mathbb{Q}$-linear map satisfying $\tau_*(tx) = t^{-1}\tau_*(x)$ and $\tau_*(\tau_*(x)) = x$ for all $x \in \mathcal{A}(K)$. As y_1 and y_2 generate $\mathcal{A}(K)$ as a $\mathbb{Q}$-module, we can write $\tau_*(y_1) = c_1y_1 + c_2y_2$ and $\tau_*(y_2) = d_1y_1 + d_2y_2$ for some $c_1, c_2, d_1, d_2 \in \mathbb{Q}$. Observe that

$$\tau_*(ty_1) = \tau_*\left(\frac{m+1}{m}y_1\right) = \frac{m+1}{m}\tau_*(y_1) = \frac{c_1(m+1)}{m}y_1 + \frac{c_2(m+1)}{m}y_2$$

and

$$t^{-1}\tau_*(y_1) = c_1t^{-1}y_1 + c_2t^{-1}y_2 = \frac{c_1 m}{m+1}y_1 + \frac{c_2(m+1)}{m}y_2.$$

As $\tau_*(ty_1) = t^{-1}\tau_*(y_1)$, it follows that $c_1 = 0$, and an analogous argument using $\tau_*(ty_2) = t^{-1}\tau_*(y_2)$ shows that $d_2 = 0$ as well. As $(\tau_*)^2 = \mathrm{Id}$, we can further conclude that $c_2d_1 = 1$. So, let $c := c_2$, and observe that we have shown that $\tau_*(y_1) = cy_2$ and $\tau_*(y_2) = \frac{1}{c}y_1$ for some nonzero $c \in \mathbb{Q}$.

We now compute $\mathcal{B}\ell_K(y_1, y_1)$, $\mathcal{B}\ell_K(y_1, y_2)$, $\mathcal{B}\ell_K(y_2, y_1)$, and $\mathcal{B}\ell_K(y_2, y_2)$, relying on Proposition 3.1. Observe that

$$(A - tA^T)^{-1} = \frac{-1}{\Delta_K(t)}\begin{pmatrix} \ell(1-t) & mt - (m+1) \\ (m+1)t - m & 0 \end{pmatrix},$$

where $\Delta_K(t) = (mt - (m+1))((m+1)t - m)$. Therefore, $\mathcal{B}\ell_K(b_1, b_1) = \frac{-\ell(1-t)^2}{\Delta_K(t)}$. Using the fact that $\mathcal{B}\ell_K(p(t)b_1, q(t)b_2) = p(t)q(t^{-1})\mathcal{B}\ell_K(b_1, b_2)$ we therefore compute:

$$\mathcal{B}\ell_K(y_1, y_1) = ((m+1)t - m)((m+1)t^{-1} - m)\frac{-\ell(1-t)^2}{\Delta_K(t)} = 0 \in \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$$

$$\mathcal{B}\ell_K(y_1, y_2) = -\ell t^{-1}(1-t)^2\frac{(m+1)t - m}{mt - (m+1)}$$

$$\mathcal{B}\ell_K(y_2, y_1) = -\ell t^{-1}(1-t)^2 \frac{mt-(m+1)}{(m+1)t-m}$$

$$\mathcal{B}\ell_K(y_2, y_2) = 0.$$

Now, let $v = (v_1, \dots, v_n)$ be any element of $\mathcal{A}(K_n) = \bigoplus^n \mathcal{A}(K)$. Each v_i can be written as $\lambda_i y_1 + \mu_i y_2$ for some $\lambda_i, \mu_i \in \mathbb{Q}$. We can then compute

$$\begin{aligned}
\mathcal{B}\ell_{K_n}(v, (\tau_n)_*(v)) &= \sum_{i=1}^n \mathcal{B}\ell_K(v_i, \tau_*(v_i)) \\
&= \sum_{i=1}^n \mathcal{B}\ell_K\left(\lambda_i y_1 + \mu_i y_2, \frac{\mu_i}{c} y_1 + \lambda_i c y_2\right) \\
&= \sum_{i=1}^n \left[\mathcal{B}\ell_K(\lambda_i y_1, \lambda_i c y_2) + \mathcal{B}\ell_K\left(\mu_i y_2, \frac{\mu_i}{c} y_1\right)\right] \\
&= \sum_{i=1}^n \left[c(\lambda_i)^2 \mathcal{B}\ell_K(y_1, y_2) + \frac{(\mu_i)^2}{c}\mathcal{B}\ell_K(y_2, y_1)\right] \\
&= -\ell t^{-1}(1-t)^2 \left[\left(c\sum_{i=1}^n (\lambda_i)^2\right)\frac{(m+1)t-m}{mt-(m+1)} + \left(\frac{\sum_{i=1}^n(\mu_i)^2}{c}\right)\frac{mt-(m+1)}{(m+1)t-m}\right].
\end{aligned}$$

As $m(m+1) \neq 0$, any multiple of either $\frac{(m+1)t-m}{mt-(m+1)}$ or $\frac{mt-(m+1)}{(m+1)t-m}$ by a nonzero rational number represents a nonzero element of $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$. Lemma 3.2 therefore implies that $\mathcal{B}\ell_{K_n}(v, (\tau_n)_*(v))$ equals 0 in $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ exactly when $c\sum_{i=1}^n (\lambda_i)^2 = 0 = \frac{1}{c}\sum_{i=1}^n (\mu_i)^2$, which occurs exactly when $\lambda_i = 0 = \mu_i$ for all $i = 1, \dots, n$, that is, exactly when $v = 0 \in \mathcal{A}(K_n)$. □

4 | EQUIVARIANT ALGEBRAIC CONCORDANCE

In this section, we define an equivariant algebraic concordance group $\mathcal{AC}^{SI}$, we define a homomorphism $\Psi : \mathcal{C}^{SI} \to \mathcal{AC}^{SI}$, and we use the equivariant algebraic concordance group to show that the kernel of the forgetful map $F : \mathcal{C}^{SI} \to \mathcal{C}$ is infinite rank.

4.1 | An equivariant slice obstruction

We begin by proving the following obstruction to equivariant sliceness. This is presumably already known to experts, but we could not find it in the literature. We remind the reader that a strongly invertible knot (K, τ) is *equivariantly slice* if there exists a slice disc D for K and an extension of τ to a locally linear, order two homeomorphism $\hat{\tau}$ of D^4 such that $D = \hat{\tau}(D)$. Unlike in our definition of concordance, we do not need to specify a direction on K.

Proposition 4.1. *Let (K, τ) be a strongly invertible knot. If (K, τ) is equivariantly slice then there exists a submodule $P \leqslant \mathcal{A}^{\mathbb{Z}}(K)$ such that the following hold.*

(1) *P is a metabolizer for the integral Blanchfield pairing, that is,*

$$P = P^{\perp} := \{y \in \mathcal{A}^{\mathbb{Z}}(K) : \mathcal{B}\ell(x, y) = 0 \textit{ for all } x \in P\}.$$

(2) *P is τ_*-invariant, that is, $\tau_*(P) = P$.*

The same holds with Λ coefficients, for $\mathcal{A}(K)$ and the rational Blanchfield pairing $\mathcal{B}\ell_K$.

Proof. Let D be a slice disc for K, and recall that $E_D := D^4 \setminus \nu(D)$ is a compact 4-manifold with $\partial E_D = M_K$. Let $P' := \ker(H_1(M_K; \mathbb{Z}[t^{\pm 1}]) \to H_1(E_D; \mathbb{Z}[t^{\pm 1}]))$, and let

$$P := \{p \in H_1(M_K; \mathbb{Z}[t^{\pm 1}]) : kp \in P' \text{ for some } k \in \mathbb{Z} \setminus \{0\}\}.$$

It is well-known [24, Theorem 2.1; 25, Theorem 2.4] that P is a metabolizer for the Blanchfield pairing, establishing item (1).

Now suppose that τ extends over D^4 to $\hat{\tau}$ with $D = \hat{\tau}(D)$. It follows that

$$\begin{aligned} P' &= \ker(H_1(M_K; \mathbb{Z}[t^{\pm 1}]) \to H_1(E_D; \mathbb{Z}[t^{\pm 1}])) = \ker(H_1(M_K; \mathbb{Z}[t^{\pm 1}]) \to H_1(E_{\hat{\tau}(D)}; \mathbb{Z}[t^{\pm 1}])) \\ &= \tau_*(\ker(H_1(M_K; \mathbb{Z}[t^{\pm 1}]) \to H_1(E_D; \mathbb{Z}[t^{\pm 1}]))) = \tau_*(P'). \end{aligned}$$

As τ_* is $\mathbb{Z}$-linear, $p \in P$ if and only if $kp \in P'$ for some $k \in \mathbb{Z} \setminus \{0\}$, if and only if $k\tau_*(p) \in P'$ (because $k\tau_*(p) = \tau_*(kp) \in \tau_*(P') = P'$), if and only if $\tau_*(p) \in P$. We have therefore established item (2), that P is τ_*-invariant.

The version with Λ coefficients is easier: we can simply take $P := \ker(H_1(M_K; \Lambda) \to H_1(E_D; \Lambda))$. □

This is an effective obstruction to equivariant sliceness. For example, when combined with Proposition 3.4 it shows the following, which proves Corollary 4.2 from the introduction. In many individual cases, we expect this could also be proven using Sakuma's η invariant, although it is not obvious how to apply that invariant to a general family of knots such as this.

Corollary 4.2. *Let (K, τ) be a genus one strongly invertible knot with nontrivial Alexander polynomial. Then (K, τ) is not equivariantly slice.*

Proof. If K is not algebraically slice then it is not even slice, so is certainly not equivariantly slice. Suppose that K is algebraically slice with nontrivial Alexander polynomial. If (K, τ) were equivariantly slice, there would be an invariant metabolizer P for the Blanchfield form, by Proposition 4.1. As $\Delta_K \neq 1$, $\mathcal{A}^{\mathbb{Z}}(K)$ is a nontrivial $\mathbb{Z}[t^{\pm 1}]$-module, so P must be nontrivial as well. Moreover, for every $x \in P$ we would have $\mathcal{B}\ell_K(x, \tau_*(x)) = 0$. But we computed in Proposition 3.4 that this holds only for $x = 0$. Thus, there is no such P. □

As noted in the introduction, the proof of Proposition 4.1 carries through identically under the weaker hypothesis that K bounds a slice disc D such that for some extension $\hat{\tau}$, one has that D and $\hat{\tau}(D)$ are isotopic rel. boundary. So, Corollary 4.2 also shows that genus one knots with nontrivial Alexander polynomials are not isotopy-equivariantly slice.

4.2 | The equivariant algebraic concordance group

Proposition 4.1 motivates the following definition, which we use to formalize the results on equivariant slicing.

Similarly to before, given a $\mathbb{Z}[t^{\pm 1}]$-module U, we write $\overline{U}$ for the same abelian group as U with the involuted $\mathbb{Z}[t^{\pm 1}]$ action, that is, $(p,u) \mapsto \overline{p} \cdot u$. Given a sesquilinear, Hermitian pairing $B \colon U \times U \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$, there is an involuted pairing $\overline{B} \colon \overline{U} \times \overline{U} \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ given by $\overline{B}(x,y) = \overline{B(x,y)}$. This is also Hermitian but has the opposite convention on the meaning of sesquilinearity, that is $\overline{B}(px,qy) = \overline{p}\overline{B}(x,y)q$.

Definition 4.3. We introduce a set and an equivalence relation which will lead to a definition of the equivariant algebraic concordance group.

(1) We consider the set of triples $(H, \mathcal{B}\ell, \tau)$, consisting of the following data.
  (i) A finitely generated $\mathbb{Z}[t^{\pm 1}]$-module H, that is $\mathbb{Z}[t^{\pm 1}]$-torsion, $\mathbb{Z}$-torsion-free, and such that $m_{1-t} \colon H \to H; x \mapsto (1-t)\cdot x$ is in fact an isomorphism.[†]
  (ii) A sesquilinear, Hermitian, nonsingular pairing $\mathcal{B}\ell \colon H \times H \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$.
  (iii) An anti-isometry $\tau \colon H \to H$ with $\tau^2 = \mathrm{Id}$. That is, $\tau \colon H \to H$ is an anti-automorphism, or in other words a $\mathbb{Z}[t^{\pm 1}]$-module isomorphism $\tau \colon H \xrightarrow{\cong} \overline{H}$, that induces an isometry between $\mathcal{B}\ell \colon H \times H \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ and $\overline{\mathcal{B}\ell} \colon \overline{H} \times \overline{H} \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$.

  We call a triple $(H, \mathcal{B}\ell, \tau)$ an *abstract equivariant Blanchfield pairing*, and remark that conditions (i) and (ii) above are exactly those needed to imply that $(H, \mathcal{B}\ell)$ arises as the (Alexander module, Blanchfield pairing) of some knot in S^3 [29].

(2) An *isometry* of abstract equivariant Blanchfield pairings $\theta \colon (H_1, \mathcal{B}\ell_1, \tau_1) \xrightarrow{\cong} (H_2, \mathcal{B}\ell_2, \tau_2)$ is an isometry $\theta \colon H_1 \to H_2$ of Blanchfield pairings such that $\theta \circ \tau_1 = \tau_2 \circ \theta$.

(3) We say that $(H, \mathcal{B}\ell, \tau)$ is *metabolic* if there is a $\mathbb{Z}[t^{\pm 1}]$-submodule $P \subseteq H$, called a *metabolizer*, such that
  (a) $P = P^{\perp} := \{y \in H : \mathcal{B}\ell(x,y) = 0 \text{ for all } x \in P\}$;
  (b) P is τ-invariant, that is, $\tau(P) = P$.

(4) The sum of two abstract equivariant Blanchfield pairings $(H_1, \mathcal{B}\ell_1, \tau_1)$ and $(H_2, \mathcal{B}\ell_2, \tau_2)$ is

$$(H_1, \mathcal{B}\ell_1, \tau_1) \oplus (H_2, \mathcal{B}\ell_2, \tau_2) := (H_1 \oplus H_2, \mathcal{B}\ell_1 \oplus \mathcal{B}\ell_2, \tau_1 \oplus \tau_2).$$

(5) We say that two abstract equivariant Blanchfield pairings $(H_1, \mathcal{B}\ell_1, \tau_1)$ and $(H_2, \mathcal{B}\ell_2, \tau_2)$ are *algebraically concordant* if there are metabolic pairings (U_1, B_1, σ_1) and (U_2, B_2, σ_2) such that there is an isometry

$$(H_1, \mathcal{B}\ell_1, \tau_1) \oplus (U_1, B_1, \sigma_1) \cong (H_2, \mathcal{B}\ell_2, \tau_2) \oplus (U_2, B_2, \sigma_2).$$

  It is immediate to see that algebraic concordance is an equivalence relation.

Remark 4.4. Does stably metabolic imply metabolic? If so, we could simplify the equivalence relation to requiring that $(H_1, \mathcal{B}\ell_1, \tau_1) \oplus (H_2, -\mathcal{B}\ell_2, \tau_2)$ is metabolic.

[†] As observed by Levine [29, Corollary 1.3], in (1)(i) the property of being $\mathbb{Z}[t^{\pm 1}]$-torsion is redundant, as it follows from the combination of finite generation and m_{1-t} being an isomorphism.

Proposition 4.5. *With respect to the given addition, the set of algebraic concordance classes of abstract equivariant Blanchfield pairings forms a group. The inverse of* $(H, \mathcal{B}\ell, \tau)$ *is* $(H, -\mathcal{B}\ell, \tau)$.

We call this group the *equivariant algebraic concordance group*, and denote it $\mathcal{AC}^{SI}$. If $(H, \mathcal{B}\ell, \tau) = 0 \in \mathcal{AC}^{SI}$ then we say that $(H, \mathcal{B}\ell, \tau)$ is *equivariantly algebraically slice*. Similarly, if a strongly invertible knot (K, τ) lies in $\ker \Psi$, then we say that (K, τ) is *equivariantly algebraically slice*.

Proof of Proposition 4.5. It is straightforward to argue that the addition is well-defined on equivalence classes, that it is associative, and that the equivalence class containing all metabolic abstract equivariant Blanchfield pairings is the identity. We need to prove that the inverse of $(H, \mathcal{B}\ell, \tau)$ is $(H, -\mathcal{B}\ell, \tau)$, or in other words that there is a metabolic pairing (U, B, σ) such that

$$(H, \mathcal{B}\ell, \tau) \oplus (H, -\mathcal{B}\ell, \tau) \oplus (U, B, \sigma)$$

is metabolic. In fact, we can take $U = 0$, and define the diagonal submodule

$$P := \{(x, x) \in H \oplus H \,:\, x \in H\}.$$

We now need to check that P is an invariant metabolizer for $\mathcal{B}\ell \oplus -\mathcal{B}\ell$. We first check that $P = P^{\perp}$. Observe that $(\mathcal{B}\ell \oplus -\mathcal{B}\ell)((x, x), (y, y)) = \mathcal{B}\ell(x, y) - \mathcal{B}\ell(x, y) = 0$ for all $x, y \in H$ and therefore for every (x, x) and (y, y) in P. Therefore, $P \subseteq P^{\perp}$. Now, let $(x, y) \in P^{\perp}$. Observe that for any $z \in H$ we have $(z, z) \in P$ and so

$$\mathcal{B}\ell(x - y, z) = \mathcal{B}\ell(x, z) - \mathcal{B}\ell(y, z) = (\mathcal{B}\ell \oplus -\mathcal{B}\ell)((x, y), (z, z)) = 0.$$

By the nonsingularity of $\mathcal{B}\ell$, we can therefore conclude that $x - y = 0$ and so $x = y$ and $(x, y) \in P$. So, $P^{\perp} \subseteq P$ as well. To see that $\tau(P) = P$, we compute that for any $(x, x) \in P$ we have $(\tau \oplus \tau)(x, x) = (\tau(x), \tau(x)) \in P$. Therefore, P is τ-invariant. This completes the proof that P is a metabolizer, and therefore completes the proof that $\mathcal{AC}^{SI}$ is a group. □

Proposition 4.6. *Taking the integral Blanchfield form of a strongly invertible knot* (K, τ) *together with the involution-induced map on the integral Alexander module* $\mathcal{A}^{\mathbb{Z}}(K)$ *gives rise to a homomorphism* $\Psi : \mathcal{C}^{SI} \to \mathcal{AC}^{SI}$.

Proof. We showed in Proposition 2.8 that the involution-induced map $\tau_* : \mathcal{A}^{\mathbb{Z}}(K) \to \mathcal{A}^{\mathbb{Z}}(K)$ is an anti-isometry of the Blanchfield pairing. Thus, we obtain an element of the codomain $\mathcal{AC}^{SI}$. We know from the ordinary algebraic concordance group that $\Psi(-K, \tau) = (\mathcal{A}^{\mathbb{Z}}(K), -\mathcal{B}\ell^{\mathbb{Z}}_K, \tau_*) = -\Psi(K, \tau)$.

We check that Ψ is well-defined. The argument is at this stage standard and purely formal. Suppose that (K_1, τ_1) and (K_2, τ_2) are equivariantly concordant. Then $(K_1 \# -K_2, \tau_1 \# -\tau_2)$ is equivariantly slice, and therefore

$$\begin{aligned}(\mathcal{A}^{\mathbb{Z}}(K_1 \# -K_2), \mathcal{B}\ell^{\mathbb{Z}}_{K_1 \# -K_2}, \tau_1 \# -\tau_2) &\cong (\mathcal{A}^{\mathbb{Z}}(K_1), \mathcal{B}\ell^{\mathbb{Z}}_{K_1}, (\tau_1)_*) \oplus (\mathcal{A}^{\mathbb{Z}}(K_2), -\mathcal{B}\ell^{\mathbb{Z}}_{K_2}, (-\tau_2)_*)\\ &= \Psi(K_1, \tau_1) \oplus -\Psi(K_2, \tau_2)\end{aligned}$$

is a metabolic form (U, B, σ) by Proposition 4.1. We also used Proposition 2.14 here. Add $\Psi(K_2, \tau_2)$ to both sides to see that

$$\Psi(K_1, \tau_1) \oplus -\Psi(K_2, \tau_2) \oplus \Psi(K_2, \tau_2) \cong (U, B, \sigma) \oplus \Psi(K_2, \tau_2).$$

On the left-hand side, $-\Psi(K_2, \tau_2) \oplus \Psi(K_2, \tau_2)$ is metabolic, as we showed in the proof of Proposition 4.5. As (U, B, σ) is also metabolic, it follows that $\Psi(K_1, \tau_1) = (\mathcal{A}^{\mathbb{Z}}(K_1), \mathcal{B}\ell^{\mathbb{Z}}_{K_1}, (\tau_1)_*)$ and $\Psi(K_2, \tau_2) = (\mathcal{A}^{\mathbb{Z}}(K_2), \mathcal{B}\ell^{\mathbb{Z}}_{K_2}, (\tau_2)_*)$ are algebraically concordant. Thus, $\Psi : \mathcal{C}^{SI} \to \mathcal{AC}^{SI}$ is a well-defined map as desired.

Finally, we know by Proposition 2.14 and the observation in the first paragraph of the proof that for every pair of strongly invertible knots (K_1, τ_1) and (K_2, τ_2), we have that

$$\Psi((K_1, \tau_1)\# - (K_2, \tau_2)) = \Psi(K_1, \tau_1) \oplus \Psi(-K_2, -\tau_2) = \Psi(K_1, \tau_1) \oplus -\Psi(K_2, \tau_2).$$

It follows that Ψ is indeed a homomorphism. □

4.3 | The kernel of *F*

We consider the forgetful map $F : \mathcal{C}^{SI} \to \mathcal{C}$. Recall that Theorem 1.3 asserts that $\ker F$ contains a subgroup of infinite rank, which is detected by the equivariant algebraic concordance group. Combining Propositions 3.4 and 4.1 implies the following.

Theorem 4.7. *Let $K_1, \dots, K_n$ be genus one algebraically slice knots with nontrivial and pairwise distinct Alexander polynomials and strong inversions τ_i. Let $a_1, \dots, a_n \in \mathbb{N}$, and let $(a_iK_i, a_i\tau_i)$ denote the a_i-fold equivariant connected sum of (K_i, τ_i). The knot $(K, \tau) = \#_{i=1}^n (a_iK_i, a_i\tau_i)$ is not equivariantly algebraically slice and is therefore not equivariantly slice.*

Proof. We work with Λ coefficients. Let $x = (x_1, \dots, x_n)$ be an element of $\mathcal{A}(K) = \oplus_{i=1}^n \mathcal{A}(a_iK_i)$. As the Alexander polynomials of $K_1, \dots, K_n$ are distinct degree 2 symmetric polynomials satisfying $|p(1)| = 1$, they are pairwise relatively prime. By the multiplicativity of Alexander polynomials under connected sum, the Alexander polynomials of $a_1K_1, \dots, a_nK_n$ are also pairwise relatively prime. By Lemma 3.2, it follows that

$$\mathcal{B}\ell_K(x, \tau_*(x)) = \sum_{i=1}^n \mathcal{B}\ell_{a_iK_i}(x_i, (a_i\tau_i)_*(x_i)) = 0$$

if and only if $Bl_{a_iK_i}(x, (a_i\tau_i)_*(x_i)) = 0$ for all $i = 1, \dots, n$. By Proposition 3.4, for each $i = 1, \dots, n$ we have that $\mathcal{B}\ell_{a_iK_i}(x_i, (a_i\tau_i)_*(x_i)) = 0$ if and only if $x_i = 0$. Therefore, $\mathcal{B}\ell_K(x, \tau_*(x)) = 0$ if and only if $x = 0$, and because $\mathcal{A}(K)$ is nontrivial it immediately follows that there is no τ_* invariant metabolizer for the Blanchfield pairing of K. Therefore, (K, τ) is not equivariantly algebraically slice and by Proposition 4.1, K is not equivariantly slice. □

It is now straightforward to prove Theorem 1.3 from the introduction, which follows from the next corollary, for example, by taking the pretzel knots $K_i := P(2i + 1, -2i - 1, 2i + 1)$.

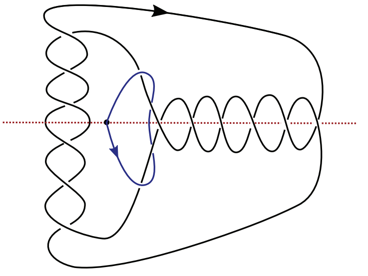

FIGURE 5 The knot K_a for $a = 3$ and the curve $\beta_1 \subseteq S^3 \setminus K$.

Corollary 4.8. *Let $\{K_i\}_{i=1}^{\infty}$ be a collection of strongly invertible genus one slice knots with nontrivial and pairwise distinct Alexander polynomials. Then the $\{K_i\}_{i=1}^{\infty}$ generate an infinite rank subgroup of* $\ker(F : \mathcal{C}^{SI} \to \mathcal{C})$ *whose image in $\mathcal{AC}^{SI}$ is also infinite rank.*

Proof. It suffices to check that for every linear combination of the K_i, $J := \#_i b_i K_i$, with $b_i \neq 0$ for only finitely many i, we have $\Psi(J) \neq 0$, and therefore J is not equivariantly slice. If $b_i \geqslant 0$ then set $a_i := b_i$ and write $K_i' := K_i$, while if $b_i < 0$ then set $a_i = -b_i$ and $K_i' := -K_i$. Then note that $J = \#_i a_i K_i'$. The K_i' have nontrivial pairwise distinct Alexander polynomials, are genus one, and are strongly invertible. Therefore, Theorem 4.7 applies to show that J is not equivariantly algebraically slice, and so J is not equivariantly slice. This shows that $\{\Psi(K_i)\}_{i=1}^{\infty}$ is an infinite rank subgroup of $\mathcal{AC}^{SI}$, and therefore that the $\{K_i\}_{i=1}^{\infty}$ generate an infinite rank subgroup of $\ker F$ as claimed. □

4.4 | Some amphichiral examples

In this subsection, we show that many order two knots in $\mathcal{C}$ map to infinite order equivariant Blanchfield pairings in $\mathcal{AC}^{SI}$, and so are also infinite order in $\mathcal{C}^{SI}$.

Let $K := K_a$ be a generalized twist 2-bridge knot corresponding to $\frac{4a^2+1}{2a}$ for some $a > 0$ (i.e., a full twists instead of 3 in each of the twist regions in Figure 5), with axis of strong inversion γ as indicated in Figure 5. Note that K_1 is the figure eight knot and that K_a is amphichiral for all $a > 0$.

Applying Seifert's algorithm to the diagram of Figure 5, we see that K has a genus one Seifert surface F and basis α_1, α_2 for $H_1(F)$ with Seifert matrix $A = \left(\begin{smallmatrix} a & 0 \\ 1 & -a \end{smallmatrix}\right)$. Let β_1, β_2 be the corresponding dual generating set for $H_1(S^3 \setminus F)$, and observe that β_1 is represented by the curve illustrated in Figure 5. Finally, note that τ preserves the given basepoint for E_{K_a} and sends β_1, considered using the basing shown as an element of $\pi_1(S^3 \setminus K)$, to $\mu^{-1}\beta_1\mu$, for a certain meridian μ of K. So, $\tau_*[\widetilde{\beta_1}] = t^{-1}\tau_*[\widetilde{\beta_1}] \in \mathcal{A}(K)$.

Following the notation of Proposition 3.1, we have that $\mathcal{A}(K)$ is generated by b_1, b_2 and has relations $a(t-1)b_1 + tb_2 = 0$ and $-b_1 - a(t-1)b_2 = 0$. Simplifying this gives that $\mathcal{A}(K) \cong \mathbb{Q}[t^{\pm 1}]/p_a(t)$, generated by b_1, where $p_a(t) = \Delta_{K_a}(t) = a^2t^2 - (2a^2 - 1)t + a^2$. Additionally, we have that $\tau_*(q(t)b_1) = t^{-1}q(t^{-1})b_1$ for all $q(t) \in \mathbb{Q}[t^{\pm 1}]$.

Lemma 4.9. *For every $a > 0$, $p_a(t)$ is irreducible over $\mathbb{Q}[t^{\pm 1}]$, and hence $\mathcal{A}(K) \cong \mathbb{Q}[t^{\pm 1}]/p_a(t)$ is a field.*

Proof. As $p_a(t)$ is a degree 2 symmetric polynomial with integer coefficients that evaluates to 1 at $t = 1$, it suffices to show that $p_a(t)$ cannot be written as $t^2 q(t)q(t^{-1})$ for any $q(t) \in \mathbb{Z}[t^{\pm 1}]$. But this follows from the fact that $p_a(-1) = 4a^2 - 1 = (2n)^2 - 1$ is never a square. $\square$

Proposition 4.10. *For every $n \in \mathbb{N}$, the knot $\#^n(K_a, \tau)$ is not equivariantly slice.*

For $a = 1, 2$ we already know using Sakuma's computations of the η invariant [41] that the knots $\#^n(K_a, \tau)$ are not equivariantly slice.

Proof. First, note that when n is odd, $\#^n K_a$ is concordant to K_a, which has an irreducible Alexander polynomial by Lemma 4.9 and hence is not slice by the Fox–Milnor criterion [20]. So, we can assume that $n = 2m$ is even.

Now let $H \leqslant \mathcal{A}(\#^n K_a)$ be a (τ_*)-invariant submodule of order $p_a(t)^m$; that is, a potential (τ_*)-invariant metabolizer for the Blanchfield pairing. As

$$\mathcal{A}(\#^n K_a) \cong \bigoplus^n \mathcal{A}(K_a) = \bigoplus^n \mathbb{Q}[t^{\pm 1}]/p_a(t),$$

we know that $H \cong (\mathbb{Q}[t^{\pm 1}]/p_a(t))^m$.

Furthermore, after rearranging our summands if necessary, the submodule $H \leqslant \mathcal{A}(K_a)$ is generated as a $\mathbb{Q}[t^{\pm 1}]/p_a(t)$-module by

$$\begin{aligned} x_1 &= (1, 0, \dots, 0, q_1^1, \dots, q_1^m) \\ x_2 &= (0, 1, 0, \dots, 0, q_2^1, \dots, q_2^m) \\ &\vdots \\ x_m &= (0, \dots, 0, 1, q_m^1, \dots, q_m^m) \end{aligned}$$

for some $q_i^j \in \mathbb{Q}[t^{\pm 1}]/p_a(t)$, $1 \leqslant i, j \leqslant m$. This relies on the fact that as $p_a(t)$ is irreducible by Lemma 4.9, $\mathbb{Q}[t^{\pm 1}]/p_a(t)$ is a field and so $\mathcal{A}(\#^n K_a) \cong (\mathbb{Q}[t^{\pm 1}]/p_a(t))^n$ is a vector space. Therefore, the existence of a generating set of this form follows from some elementary linear algebra.

Write each $q_1^j = c_j + d_j t$ for some $c_j, d_j \in \mathbb{Q}$. Observe that

$$\begin{aligned} (\#^n \tau)_*(x_1) &= (\tau_*(1), \tau_*(0), \dots, \tau_*(0), \tau_*(q_1^1), \dots, \tau_*(q_1^m)) \\ &= (\tau_*(1), \tau_*(0), \dots, \tau_*(0), \tau_*(c_1 + d_1 t), \dots, \tau_*(c_m + d_m t)) \\ &= (t^{-1}, 0, \dots, 0, c_1 t^{-1} + d_1 t^{-2}, \dots, c_m t^{-1} + d_m t^{-2}). \end{aligned}$$

As $(\#^n \tau)_*(x_1) \in H$, we can write $(\#^n \tau)_*(x_1) = \sum_{i=1}^n r_i(t) x_i$ for some $r_i(t) \in \mathbb{Q}[t^{\pm 1}]/p_a(t)$. Considering our expressions for $x_1, \dots, x_m$ and for $(\#^n \tau)_*(x_1)$ and looking at the first m coordinates, we obtain that $r_1(t) = t^{-1}$ and $r_2(t) = \cdots = r_m(t) = 0$. Therefore, we have for each $j = 1, \dots, m$ that $c_j t^{-1} + d_j t^{-2} = t^{-1}(c_j + d_j t)$ in $\mathbb{Q}[t^{\pm 1}]/p_a(t)$. It follows that $d_1 = \cdots = d_m = 0$.

So, $x_1 = (1, 0, \dots, 0, c_1, c_2, \dots, c_m)$ for some $c_i \in \mathbb{Q}$. But we can now show that $\mathcal{B}\ell_{\#^n K_a}(x_1, x_1) \neq 0$, and hence that H is not a metabolizer:

$$\mathcal{B}\ell_{\#^n K_a}(x_1, x_1) = \mathcal{B}\ell_{K_a}(1, 1) + \sum_{i=1}^m \mathcal{B}\ell_{K_a}(c_i, c_i) = \left(1 + \sum_{i=1}^m c_i^2\right) \mathcal{B}\ell_{K_a}(1, 1).$$

As $\mathcal{A}(K) \cong \mathbb{Q}[t^{\pm 1}]/p_a(t)$ and $\mathcal{B}\ell_{K_a}$ is nondegenerate, we have that $\mathcal{B}\ell_{K_a}(1,1)$ is nonzero in $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$. (Of course, we could also directly compute this using the Seifert matrix and Proposition 3.1.) Therefore, as $1 + \sum_{i=1}^{m} c_i^2$ is a nonzero rational number, $\mathcal{B}\ell_{\#^n K_a}(x_1, x_1)$ is nonzero in $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ as well. □

5 | A LOWER BOUND ON THE EQUIVARIANT 4-GENUS

Now we switch our attention to proving the lower bound from Theorem 1.2, which will lead to the proof of Theorem 1.1 when combined with the computation in Proposition 3.4.

5.1 | Construction of the 4-manifold Z and its properties

As in the proof of Proposition 4.1, we extensively consider the kernel of the inclusion induced map $H_1(E_K;\Lambda) \to H_1(E_F;\Lambda)$, where F is some locally flat surface in D^4 with boundary K. However, it will simplify our arguments to work with the closed 3-manifold M_K and an associated 4-manifold Z with $\partial Z = M_K$ instead.

Proposition 5.1. *Let (K,τ) be a strongly invertible knot in S^3 bounding a genus g surface F in D^4. There exists a 4-manifold Z with boundary M_K such that the following hold.*

(1) *The inclusion-induced map $i_* : H_1(M_K;\mathbb{Z}) \to H_1(Z;\mathbb{Z})$ is an isomorphism, and so we can consider $H_1(Z;\Lambda)$ and $H_1(Z, M_K;\Lambda)$ as usual.*
(2) *$H_1(Z, M_K;\Lambda)$ and $H_1(Z;\Lambda)$ are torsion.*
(3) *The free part of $H_2(Z;\Lambda)$ has rank $2g$.*
(4) *The inclusion-induced map $i_* : H_1(E_K;\Lambda) \to H_1(M_K;\Lambda)$ is an isomorphism under which $\ker(H_1(E_K;\Lambda) \to H_1(E_F;\Lambda))$ is mapped to $\ker(H_1(M_K;\Lambda) \to H_1(Z;\Lambda))$.*
(5) *If τ extends to an involution $\widehat{\tau}$ on D^4 such that $F = \widehat{\tau}(F)$, then $H := \ker(H_1(M_K;\Lambda) \to H_1(Z;\Lambda))$ is invariant, that is, $\tau_*(H) = H$.*

We remark that items (1), (2), and (3) of Proposition 5.1 are reasonably standard and have appeared in the literature before; see, for example, [38, section 4; 10, Proposition 9.1]. We include the proof of these items here for completeness.

For the proof of Proposition 5.1, we will need the following special case of [11, Propositions 2.9 and 2.11].

Proposition 5.2. *Let X be a space with the homotopy type of a finite CW complex, and let $\phi : \pi_1(X) \to \mathbb{Z}$ be a nontrivial representation. Then $H_0(X;\mathbb{Q}(t)) = 0$ and $\dim_{\mathbb{Q}(t)} H_1(X;\mathbb{Q}(t)) \leqslant b_1(X) - 1$.*

Proof of Proposition 5.1. Define

$$Z := (D^4 \setminus \nu(F)) \bigcup_{S^1 \times F} (S^1 \times H),$$

where H is a genus g handlebody with boundary $\partial H = F \cup D^2$. To make this gluing we choose a framing of the normal bundle of F in D^4 such that for each simple closed curve $\alpha \subseteq F$, the curve $\alpha \times \{1\} \subseteq S^1 \times F \subseteq D^4 \setminus \nu F$ is null-homologous. There is also a choice of precisely which handlebody H we choose to fill $F \cup D^2$. We make an arbitrary choice here, as this does not affect the homological properties of Z that we will use. A Mayer–Vietoris argument establishes item (1), as well as the fact that $H_2(Z;\mathbb{Z}) \cong \mathbb{Z}^{2g}$. We note for later use that item (1) also implies that $H_3(Z;\mathbb{Q}) \cong H^1(Z, M_K;\mathbb{Q}) \cong \operatorname{Hom}(H_1(Z, M_K;\mathbb{Q}),\mathbb{Q}) \cong 0$, and so $\chi(Z) = 1 - 1 + 2g + 0 + 0 = 2g$.

To establish items (2) and (3), by the flatness of $\mathbb{Q}(t)$ as a Λ-module it suffices to show that $H_1(Z, M_K;\mathbb{Q}(t)) = 0$ and $H_2(Z;\mathbb{Q}(t)) \cong \mathbb{Q}(t)^{2g}$. By Proposition 5.2, we have that $H_i(M_K,\mathbb{Q}(t)) = 0 = H_i(Z;\mathbb{Q}(t))$ for $i = 0,1$. It follows from the long exact sequence of (Z, M_K) with $\mathbb{Q}(t)$ coefficients that $H_1(Z, M_K;\mathbb{Q}(t)) = 0$ as desired for item (2). Item (3) will now quickly follow from an Euler characteristic computation for Z using $\mathbb{Q}(t)$ coefficients. We already know $H_0(Z;\mathbb{Q}(t)) = 0 = H_1(Z,\mathbb{Q}(t))$. Additionally, for $i = 3,4$ we have that $H_i(Z;\mathbb{Q}(t)) = H^i(Z;\mathbb{Q}(t)) = H_{4-i}(Z, M_K;\mathbb{Q}(t)) = 0$, where the first equality comes from universal coefficients and the second from Poincaré duality. We now obtain item (3), as

$$\begin{aligned} 2g &= b_0^{\mathbb{Q}(t)}(Z) - b_1^{\mathbb{Q}(t)}(Z) + b_2^{\mathbb{Q}(t)}(Z) - b_3^{\mathbb{Q}(t)}(Z) + b_4^{\mathbb{Q}(t)}(Z) \\ &= 0 - 0 + \dim H_2(Z;\mathbb{Q}(t)) - 0 + 0. \end{aligned}$$

We now wish to establish item (4). Recall that $M_K = E_K \cup_{T^2} (S^1 \times D^2)$. A direct computation gives us that $H_1(S^1 \times D^2;\Lambda) = 0$ and $H_1(T^2;\Lambda) \cong \mathbb{Q}[t^{\pm 1}]/(t-1)$, with generator given by a class represented by a 0-framed longitude of K. As discussed in Definition 2.2, the 0-framed longitude of K is a double commutator in $\pi_1(E_K)$, and hence maps to the zero element in $H_1(E_K;\Lambda)$. So, the inclusion induced map $H_1(T^2;\Lambda) \to H_1(E_K;\Lambda)$ is the 0-map, and by considering the Mayer–Vietoris sequence for M_K we have that $i_* : H_1(E_K;\Lambda) \to H_1(M_K;\Lambda)$ is an isomorphism. Now recall that $Z = E_F \cup_{S^1 \times F} (S^1 \times H)$, where H is a genus g handlebody with $\partial H = F \cup D^2$. This decomposition is compatible with that of M_K, and so we obtain the following commutative diagram, where all maps are induced by inclusion:

$$\begin{tikzcd} H_1(E_K;\Lambda) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_1(M_K;\Lambda) = H_1(E_K \cup (S^1 \times D^2);\Lambda) \arrow[d, "g_*"] \\ H_1(E_F;\Lambda) \arrow[r, "j_*"] & H_1(Z;\Lambda) = H_1(E_F \cup (S^1 \times H);\Lambda). \end{tikzcd}$$

We wish to show that $\ker(g_*) = i_*(\ker(f_*))$. One containment is immediate: for $y \in i_*(\ker(f_*))$ write $y = i_*(x)$ for $x \in \ker(f_*)$ and observe that $g_*(y) = g_* i_*(x) = j_* f_*(x) = j_*(0) = 0$, that is $y \in \ker(g_*)$.

Now let $y \in \ker(g_*)$ and, recalling that i_* is a isomorphism, let $x \in H_1(E_K;\Lambda)$ be such that $i_*(x) = y$ in order to show that $f_*(x) = 0$. As $j_* f_*(x) = g_* i_*(x) = g_*(y) = 0$, we certainly have that $f_*(x)$ is in $\ker(j_*)$. Now consider the following portion of the Mayer–Vietoris sequence for Z:

$$H_1(S^1 \times F;\Lambda) \to H_1(E_F;\Lambda) \oplus H_1(S^1 \times H;\Lambda) \to H_1(Z;\Lambda).$$

Here we use the Λ coefficient systems on the subsets of Z that appear, namely E_F, $S^1 \times F$, and $S^1 \times H$, induced from the Λ coefficient system on Z by the inclusions. As the sequence is exact and $f_*(x)$ maps to 0 in $H_1(Z;\Lambda)$, we can conclude that $f_*(x) \in \operatorname{Im}(H_1(S^1 \times F;\Lambda) \to H_1(E_F;\Lambda))$.

One can compute directly that

$$H_1(S^1 \times F; \Lambda) = H_1(\mathbb{R} \times F; \mathbb{Q}) \cong (\lambda/(t-1))^{2g(F)},$$

and hence that $f_*(x)$ is annihilated by $t-1$. But as the order of $H_1(E_K; \Lambda)$ is $\Delta_K(t)$, it is also true that $f_*(x)$ is annihilated by $\Delta_K(t)$. We assert that $\Delta_K(t)$ and $t-1$ are relatively prime. To see this, note that, up to powers of t, any common divisor p must be constant or linear. If p is linear and divides $(t-1)$ then up to units $p = t-1$, and so $t-1 \mid \Delta_K(t)$. Evaluating at $t = 1$ yields a contradiction $\pm 1 = \Delta_K(1) = 0$. So, p cannot be linear, and p is a nonzero constant α, that is, a unit in Λ. Therefore, $\Delta_K(t)$ and $t-1$ are relatively prime as asserted. Thus, as Λ is a Euclidean domain, there are $r, s \in \Lambda$ such that $r\Delta_K(t) + s(t-1) = 1$. We therefore have as desired that

$$f_*(x) = (r\Delta_K(t) + s(t-1))f_*(x) = r(\Delta_K(t)f_*(x)) + s((t-1)f_*(x)) = 0.$$

This completes the proof of item (4).

To prove item (5), suppose that τ extends to an involution $\hat{\tau}$ on D^4 such that $F = \hat{\tau}(F)$. It follows immediately that

$$\ker(H_1(E_K; \Lambda) \to H_1(E_F; \Lambda)) = \ker(H_1(E_K; \Lambda) \to H_1(E_{\hat{\tau}(F)}; \Lambda)).$$

Note that for any surface G in D^4 with $\partial G = K$ we have

$$\ker(H_1(E_K; \Lambda) \to H_1(E_{\hat{\tau}(G)}; \Lambda)) = \tau_*(\ker(H_1(E_K; \Lambda) \to H_1(E_G; \Lambda))).$$

Therefore, $\ker(H_1(E_K; \Lambda) \to H_1(E_F; \Lambda))$ is τ_*-invariant. However, by item (4) we know that $\ker(H_1(E_K; \Lambda) \to H_1(E_F; \Lambda))$ is identified with $\ker(H_1(M_K; \Lambda) \to H_1(Z; \Lambda))$ via the inclusion induced map, which is compatible with τ_*, and so we get our desired result. □

5.2 | Blanchfield forms and generating rank

The proof of the next proposition is closely related to a standard argument, see, for example, [11, Theorem 4.4], but as we need a slight variation we give the details.

Proposition 5.3. *Let K be a knot in S^3 with zero surgery M_K, and suppose Z is a 4-manifold with $\partial Z = M_K$ such that $i_* : H_1(M_K; \mathbb{Z}) \to H_1(Z; \mathbb{Z})$ is an isomorphism. Suppose that $H_1(Z; \Lambda)$ is Λ-torsion. Then for every $x \in TH_2(Z, M_K; \Lambda)$ and every $y \in \ker(i_* : H_1(M_K; \Lambda) \to H_1(Z; \Lambda))$ we have $\mathcal{B}\ell(\partial x, y) = 0$.*

Proof. Let $\partial : H_2(Z, M_K; \Lambda) \to H_1(M_K; \Lambda)$ be the connecting map in the long exact sequence of the pair (Z, M_K) with Λ-coefficients. Restricting this to the torsion submodule $TH_2(Z, M_K; \Lambda)$ gives a map $\partial|_T : TH_2(Z, M_K; \Lambda) \to TH_1(M_K; \Lambda) = H_1(M_K; \Lambda)$, recalling that $H_1(M_K; \Lambda)$ (and $H_1(Z; \Lambda)$, for later reference) are Λ-torsion.

We therefore obtain the following commutative diagram:

$$\begin{tikzcd}
TH_2(Z, M_K; \Lambda) \arrow[r, "\partial|_T"] \arrow[d, "PD_Z^{-1}"] \arrow[ddd, bend right, "\beta"'] & H_1(M_K; \Lambda) \arrow[r, "i_*"] \arrow[d, "PD_{M_K}^{-1}"'] \arrow[ddd, bend left, "\Theta"] & H_1(Z; \Lambda) \\
TH^2(Z; \Lambda) \arrow[r, "i^*"] \arrow[d, "B_Z^{-1}"] & H^2(M_K; \Lambda) \arrow[d, "B_{M_K}^{-1}"'] & \\
H^1(Z; Q/\Lambda) \arrow[r, "i^*"] \arrow[d, "\kappa_Z"] & H^1(M_K; Q/\Lambda) \arrow[d, "\kappa_{M_K}"'] & \\
\overline{\mathrm{Hom}}(H_1(Z; \Lambda), Q/\Lambda) \arrow[r, "i_*^\wedge"] & \overline{\mathrm{Hom}}(H_1(M; \Lambda), Q/\Lambda). &
\end{tikzcd}$$

The top row is not necessarily exact; we remark to the experts that this is a key difference between our setting and that of [11]. However, we do have that $\mathrm{Im}(\partial|_T) \subseteq \ker(i_*)$, as

$$H_2(Z, M_K; \Lambda) \xrightarrow{\partial} H_1(M_K; \Lambda) \xrightarrow{i_*} H_1(Z; \Lambda)$$

is exact. Moreover, as all of the vertical maps are natural, the diagram commutes. This is straightforward for the Bockstein and universal coefficients, while [5, Theorem IV.9.2] shows that the top square commutes.

Now let $x \in TH_2(Z, M_K; \Lambda)$ and $y \in \ker(i_* : H_1(M_K; \Lambda) \to H_1(Z; \Lambda))$. We therefore have that

$$\mathcal{B}\ell(y, \partial x) = \Theta(\partial x)(y) = i_*^\wedge(\beta(x))(y) = \beta(x)(i_*(y)) = \beta(x)(0) = 0.$$

The first equality comes from the definition of $\mathcal{B}\ell$, the second equality from the commutativity of the diagram, the third equality from the definitional relationship between $i_*^\wedge$ and i_*, and the last from our assumption on y. We are now done, as

$$\mathcal{B}\ell(\partial x, y) = \overline{\mathcal{B}\ell(y, \partial x)} = \overline{0} = 0. \qquad \square$$

To effectively apply Proposition 5.1, we will need to show that $\partial(TH_2(Z, M_K; \Lambda))$ has large generating rank. It will be useful to have the following facts about the generating rank of finitely generated modules over PIDs, which follow from the fundamental theorem of finitely generated modules over PIDs; see also [34, Lemma 4.1]. Recall that the generating rank of a finitely generated module A over a PID S is the minimal number of elements needed to generate A as an S-module.

Proposition 5.4. *Let A, B be finitely generated modules over a PID S.*

(1) *If $A \subseteq B$ then* $\mathrm{g\text{-}rk}\, A \leqslant \mathrm{g\text{-}rk}\, B$.
(2) *If $f : A \to B$ is a map of S-modules, then*

$$\mathrm{g\text{-}rk}\, \mathrm{Im}(f) \leqslant \mathrm{g\text{-}rk}\, A \leqslant \mathrm{g\text{-}rk}\, \mathrm{Im}(f) + \mathrm{g\text{-}rk}\, \ker(f).$$

The next proposition is one of the key technical facts on generating ranks.

Proposition 5.5. *Let Z be a compact, oriented 4-manifold with boundary $\partial Z = M_K$ such that $i_* : H_1(M_K;\mathbb{Z}) \to H_1(Z;\mathbb{Z})$ is an isomorphism. Let n be the Λ-rank of $H_2(Z;\Lambda)$ that is, the free part of $H_2(Z;\Lambda)$ is isomorphic to Λ^n. Assume that $H_1(Z,M_K;\Lambda)$ is torsion. Then the generating rank of $\partial(TH_2(Z,M_K;\Lambda))$ is at least $\frac{1}{2}\operatorname{g-rk}\mathcal{A}(K) - n$.*

For the proof of Proposition 5.5, we will need the following result from our article [10] with Jae Choon Cha.

Lemma 5.6 [10, Lemma 7.5]. *Let X be a compact, oriented 4-manifold with boundary $\partial X = Y$. Let S be a PID, and suppose there is a representation Φ of the fundamental group of Y into $\operatorname{Aut}(S)$ that extends over X. Consider the long exact sequence of the pair (X,Y):*

$$\cdots \to H_2(X;S) \xrightarrow{j_2} H_2(X,Y;S) \xrightarrow{\partial} H_1(Y;S) \xrightarrow{i_1} H_1(X;S) \xrightarrow{j_1} H_1(X,Y;S) \to \cdots$$

If $H_1(X,Y;S)$ is torsion, then $\ker(j_1|_T)$ and $\operatorname{coker}(j_2|_T)$ are isomorphic as S-modules.

Proof of Proposition 5.5. For the entirety of this proof, all homology is by default taken with coefficients in Λ. As Λ is a PID, we can choose an isomorphism $H_2(Z) \cong \Lambda^n \oplus TH_2(Z)$. Noting that the free part of $H_2(Z,M_K)$ must also have rank n by duality and universal coefficients, we also choose an isomorphism $H_2(Z,M_K) \cong \Lambda^n \oplus TH_2(Z,M_K)$. This allows us to decompose the long exact sequence of the pair (Z,M_K) as follows:

$$\begin{tikzcd}
\Lambda^n \arrow[r, "j_2^a"] \arrow[ddr, "j_2^b"] & \Lambda^n \arrow[dr, "\partial^a"] & & & & \\
\oplus & \oplus & H_1(M_K) \arrow[r, "i_1"] & H_1(Z) \arrow[r, "j_1"] & H_1(Z,M_K) \arrow[r] & 0. \\
TH_2(Z) \arrow[r, "j_2|_T"] & TH_2(Z,M_K) \arrow[ur, "\partial|_T"'] & & & &
\end{tikzcd}$$

As $H_1(M_K)$ and $H_1(Z,M_K)$ are both Λ-torsion, the former because this holds for all knots, and that latter by assumption, it follows that $H_1(Z)$ must be torsion as well. We will use this later to conclude that $\ker(j_1|_T) = \ker(j_1)$.

Now define $k := \operatorname{g-rk}\operatorname{Im}(\partial|_T)$, and let $x_1,\dots,x_k \in TH_2(Z,M_K)$ be elements whose images $\partial|_T(x_1),\dots,\partial|_T(x_k)$ under $\partial|_T$ generate $\operatorname{Im}(\partial|_T)$. As $\operatorname{g-rk}\Lambda^n = n$, there exist $y_1,\dots,y_n \in TH_2(Z,M_K)$ that generate $\operatorname{Im}(j_2^b)$ as a Λ-module. Let $z_1,\dots,z_\ell \in TH_2(Z,M_K)$ generate $\operatorname{Im}(j_2|_T)$ as a Λ-module for some $\ell \in \mathbb{N}$.

We claim that $x_1,\dots,x_k,y_1,\dots,y_n,z_1,\dots,z_\ell$ generate $TH_2(Z,M_K)$. Let v be an arbitrary element of $TH_2(Z,M_K)$. As $\partial|_T(x_1),\dots,\partial|_T(x_k)$ generate $\operatorname{Im}(\partial|_T)$, there exist $p_1,\dots,p_k \in \Lambda$ such that

$$\partial|_T(v) = \sum_{i=1}^{k} p_i \partial|_T(x_i) = \partial|_T\left(\sum_{i=1}^{k} p_i x_i\right).$$

Therefore, $w := v - \sum_{i=1}^{k} p_i x_i$ is an element of $\ker(\partial|_T)$. As $w \in TH_2(Z,M_K)$, we have that $\partial(w) = \partial|_T(w) = 0$, and so w is an element of

$$\ker(\partial) \cap TH_2(Z,M_K) = \operatorname{Im}(j_2) \cap TH_2(Z,M_K).$$

Now we assert that $\mathrm{Im}(j_2) \cap TH_2(Z, M_K) \subseteq \mathrm{Im}(j_2^b) + \mathrm{Im}(j_2|_T)$. Assuming this, we can write $w = \sum_{i=1}^n q_i y_i + \sum_{i=1}^{\ell} r_i z_i$ for some $q_i, r_i \in \Lambda$, thereby establishing that $x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_\ell$ generate $TH_2(Z, M_K)$. To complete the proof of this claim we now argue that

$$\mathrm{Im}(j_2) \cap TH_2(Z, M_K) \subseteq \mathrm{Im}(j_2^b) + \mathrm{Im}(j_2|_T)$$

as follows. Let $u \in \mathrm{Im}\, j_2$, that is $u = j_2(s) = j_2(s_f, s_T)$ for $s_f \in \Lambda^n$ and $s_T \in TH_2(Z)$. More precisely, $u = j_2(s_f, s_T) = j_2^a(s_f) + j_2^b(s_T) + j_2|_T(s_T)$. If in addition $u \in TH_2(Z, M_K)$, then $j_2^a(s_f) = 0$ and so indeed $u \in \mathrm{Im}(j_2^b) + \mathrm{Im}(j_2|_T)$.

It follows that the equivalence classes of $x_1, \ldots, x_k, y_1, \ldots, y_n$ generate $TH_2(Z, M_K)/\,\mathrm{Im}(j_2|_T)$, and hence that $\mathrm{g\text{--}rk}\,\mathrm{coker}(j_2|_T) \leqslant n + k$. By Lemma 5.6, using the hypothesis that $H_1(Z, M_K)$ is torsion in order to apply the lemma, this implies that $\mathrm{g\text{--}rk}\,\mathrm{coker}(j_2|_T) = \mathrm{g\text{--}rk}\,\mathrm{ker}(j_1|_T)$, and therefore we have

$$\mathrm{g\text{--}rk}\,\mathrm{ker}(j_1) = \mathrm{g\text{--}rk}\,\mathrm{ker}(j_1|_T) = \mathrm{g\text{--}rk}\,\mathrm{coker}(j_2|_T) \leqslant n + k.$$

For the first equality we used that $H_1(Z) = TH_1(Z)$, as observed above. Also note that

$$\ker(i_1) = \mathrm{Im}(\partial) = \mathrm{Im}(\partial^a) + \mathrm{Im}(\partial|_T),$$

and so by recalling that ∂^a has domain Λ^n we obtain that

$$\mathrm{g\text{--}rk}\,\ker(i_1) \leqslant \mathrm{g\text{--}rk}\,\mathrm{Im}(\partial^a) + \mathrm{g\text{--}rk}\,\mathrm{Im}(\partial|_T) \leqslant \mathrm{g\text{--}rk}\,\Lambda^n + \mathrm{g\text{--}rk}\,\mathrm{Im}(\partial|_T) \leqslant n + k.$$

Now combine Proposition 5.4(2), exactness, and the previous two inequalities ($\mathrm{g\text{--}rk}\,\ker(j_1) \leqslant n + k$ and $\mathrm{g\text{--}rk}\,\ker(i_1) \leqslant n + k$), to obtain

$$\begin{aligned}\mathrm{g\text{--}rk}\,\mathcal{A}(K) = \mathrm{g\text{--}rk}\,H_1(M_K) &\leqslant \mathrm{g\text{--}rk}\,\ker(i_1) + \mathrm{g\text{--}rk}\,\mathrm{Im}(i_1) = \mathrm{g\text{--}rk}\,\ker(i_1) + \mathrm{g\text{--}rk}\,\ker(j_1)\\ &\leqslant (n + k) + (n + k) = 2n + 2k.\end{aligned}$$

We therefore have that

$$\mathrm{g\text{--}rk}\,\mathrm{Im}(\partial|_T) = k \geqslant (\mathrm{g\text{--}rk}\,\mathcal{A}(K) - 2n)/2 = \frac{1}{2}\,\mathrm{g\text{--}rk}\,\mathcal{A}(K) - n,$$

as desired. □

5.3 | Proofs of Theorems 1.1 and 1.2

We are now ready to prove these two theorems.

Proof of Theorem 1.2. Recall that (K, τ) is a strongly invertible knot and k is by definition the maximal generating rank of any submodule P of $\mathcal{A}(K)$ satisfying $\mathcal{B}\ell_K(x, y) = 0 = \mathcal{B}\ell_K(x, \tau_*(y))$ for all $x, y \in P$. Now suppose that K bounds a genus g surface F in D^4 such that the involution τ on S^3 extends to an involution $\widehat{\tau}$ on D^4 such that $\widehat{\tau}(F) = F$. We wish to show that $g \geqslant \frac{\mathrm{g\text{--}rk}\,\mathcal{A}(K) - 2k}{4}$.

Let Z be as in Proposition 5.1, and consider the following portion of the long exact sequence of (Z, M_K) with Λ-coefficients:

$$\cdots \to H_2(Z, M_K; \Lambda) \xrightarrow{\partial} H_1(M_K; \Lambda) \xrightarrow{i_*} H_1(Z; \Lambda) \to \cdots.$$

Define $Q := \partial(TH_2(Z, M_K; \Lambda)) \subseteq H_1(M_K; \Lambda)$.

Our first claim is that $\mathcal{B}\ell_K(x, y) = 0 = \mathcal{B}\ell_K(x, \tau_*(y))$ for all $x, y \in Q$. So, let $x, y \in Q$ be given. As $Q \subseteq \mathrm{Im}(\partial|_T) \subseteq \ker(i_*)$, Proposition 5.3 implies that $\mathcal{B}\ell_K(x, y) = 0$. Additionally, $y \in \ker(i_*)$ implies that $\tau_*(y) \in \ker(i_*)$ as well by Proposition 5.1(5). Thus, $\mathcal{B}\ell_K(x, \tau_*(y)) = 0$ too. We conclude that $\mathrm{g\text{-}rk}\, Q \leqslant k$, by definition of k.

By Proposition 5.1(2) and (3), we have that $H_1(Z, M_K; \Lambda)$ is torsion and the free part of $H_2(Z; \Lambda)$ has rank $2g$. Therefore, Proposition 5.5 implies that $\mathrm{g\text{-}rk}\, Q \geqslant \frac{1}{2}\, \mathrm{g\text{-}rk}\, \mathcal{A}(K) - 2g$. We therefore have that

$$k \geqslant \mathrm{g\text{-}rk}\, Q \geqslant \frac{1}{2}\, \mathrm{g\text{-}rk}\, \mathcal{A}(K) - 2g$$

or, rewriting,

$$g \geqslant \frac{\mathrm{g\text{-}rk}\, \mathcal{A}(K) - 2k}{4}. \qquad \square$$

Finally, Theorem 1.1 is an immediate consequence of the following slightly stronger result.

Theorem 5.7. *Let $J_1, \ldots, J_n$ denote genus one strongly invertible knots with pairwise distinct and nontrivial Alexander polynomials. Pick a strong inversion τ_i on J_i for each $i = 1, \ldots, n$ and choose $a_1, \ldots, a_n \in \mathbb{N}$. Letting $\#^{a_i}(J_i, \tau_i)$ denote the a_i-fold connected sum of (J_i, τ_i), define $(J, \tau) := \#_{i=1}^n(\#^{a_i}(J_i, \tau_i))$. Then the equivariant 4-genus of (J, τ) is at least $\frac{1}{4}\max(a_1, \ldots, a_n)$.*

Proof. First, observe that

$$\mathrm{g\text{-}rk}\, \mathcal{A}(J) = \mathrm{g\text{-}rk} \bigoplus_{i=1}^n \mathcal{A}(J_i)^{a_i} = \mathrm{g\text{-}rk} \bigoplus_{i=1}^n \left(\mathbb{Q}[t^{\pm 1}]/\Delta_{J_i}(t)\right)^{a_i} = \max\{a_1, \ldots, a_n\},$$

where the last equality uses the fact that $\Delta_{J_1}(t), \ldots, \Delta_{J_n}(t)$ are pairwise distinct, degree 2, and symmetric, hence pairwise relatively prime.

It remains to show that the only element $x \in \mathcal{A}(J)$ with $\mathcal{B}\ell_J(x, \tau_*(x)) = 0$ is the trivial element, and our result will follow by Theorem 1.2. So, write $x = (x_i)_{i=1}^n$, where each $x_i \in \mathcal{A}(\#^{a_i} J_i)$, and observe that we can write $\mathcal{B}\ell_{\#^{a_i} J_i}(x_i, (\tau_i)_*(x_i)) = \frac{p_i(t)}{\Delta_{J_i}(t)}$ for some $p_i(t) \in \mathbb{Q}[t^{\pm 1}]$. (This follows from the general fact that $\mathcal{B}\ell_K$ takes values in $\frac{1}{\Delta_K(t)}\Lambda/\Lambda$.) So,

$$\mathcal{B}\ell_J(x, \tau_*(x)) = \sum_{i=1}^n \mathcal{B}\ell_{\#^{a_i} J_i}(x_i, (\tau_i)_*(x_i)) = \sum_{i=1}^n \frac{p_i(t)}{\Delta_{J_i}(t)}.$$

As all the $\Delta_{J_i}(t)$ are relatively prime, by Lemma 3.2 this expression is trivial in $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ only when $\mathcal{B}\ell_{\#^{a_i} J_i}(x_i, (\tau_i)_*(x_i)) = \frac{p_i(t)}{\Delta_{J_i}(t)}$ vanishes for all $i = 1, \ldots, n$. But by Proposition 3.4 applied to each J_i, this occurs only when $x_i = 0$ for all $i = 1, \dots, n$, that is when $x = 0$. Therefore, $k = 0$ in Theorem 1.2, so

$$\widetilde{g}_4(J) \geqslant \frac{\text{g-rk}\,\mathcal{A}(J)}{4} = \frac{1}{4}\max(a_1, \dots, a_n),$$

as desired. $\square$

ACKNOWLEDGEMENTS

The authors are grateful to Irving Dai, Abhishek Mallick, and Matthew Stoffregen for interesting conversations with Mark Powell about their work [15], which motivated this project. Mark Powell is also grateful to Maciej Borodzik and Wojciech Politarczyk for interesting conversations about strongly invertible knots. The authors thank Chuck Livingston for helpful comments on a draft of this piece. They also thank an anonymous referee for a careful reading and pertinent comments. Mark Powell was partially supported by EPSRC New Investigator Grant EP/T028335/1 and EPSRC New Horizons Grant EP/V04821X/1.

REFERENCES

1. A. Alfieri and K. Boyle, *Strongly invertible knots, invariant surfaces, and the Atiyah-Singer signature theorem*, arXiv:2109:09915, 2021.
2. K. Boyle and A. Issa, *Equivariant 4-genera of strongly invertible and periodic knots*, J. Topol. **15** (2022) 1635–1674.
3. Blanchfield, R. C., *Intersection theory of manifolds with operators with applications to knot theory*, Ann. Math. (2) **65** (1957), 340–356.
4. G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York-London, 1972.
5. G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer, New York, 1997. (Corrected third printing of the 1993 original.)
6. A. Casson and C. Gordon, *On slice knots in dimension three*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, 1978, pp. 39–53.
7. J. C. Cha, *Primary decomposition in the smooth concordance group of topologically slice knots*, Forum Math. Sigma **9** (2021), 37.
8. T. D. Cochran, S. Harvey, and C. Leidy, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. **13** (2009), no. 3, 1419–1482.
9. T. D. Cochran, S. Harvey, and C. Leidy, *Primary decomposition and the fractal nature of knot concordance*, Math. Ann. **351** (2011), no. 2, 443–508.
10. J. C. Cha, A. N. Miller, and M. Powell, *Two-solvable and two-bipolar knots with large four-genera*, Math. Res. Lett. **28** (2021), no. 2, 331–382 (English).
11. T. D. Cochran, K. E. Orr, and P. Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.
12. A. Conway and M. Powell, *Characterisation of homotopy ribbon discs*, Adv. Math. **391** (2021), 29.
13. S. E. Cappell and J. L. Shaneson, *Topological knots and knot cobordism*, Topology **12** (1973), 33–40.
14. I. Dai, M. Hedden, and A. Mallick, *Corks, involutions, and Heegaard Floer homology*. To appear in Journal of the EMS, arXiv:2002.02326, 2022.

15. I. Dai, A. Mallick, and M. Stoffregen, *Equivariant knots and knot Floer homology*, arXiv:2201.01875, 2022.
16. A. Di Prisa, *The equivariant concordance group is not abelian*, arXiv:2207.04985, 2022.
17. H. Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology Appl. **63** (1995), no. 3, 257–262 (English).
18. M. Farber, *Hermitian forms on link modules*, Comment. Math. Helv. **66** (1991), no. 2, 189–236.
19. S. Friedl, C. Leidy, M. Nagel, and M. Powell, *Twisted Blanchfield pairings and decompositions of 3-manifolds*, Homology Homotopy Appl. **19** (2017), no. 2, 275–287.
20. R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka Math. J. **3** (1966), 257–267.
21. S. Friedl and M. Powell, *A calculation of Blanchfield pairings of 3-manifolds and knots*, Mosc. Math. J. **17** (2017), no. 1, 59–77 (English).
22. M. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.
23. B. D. Franklin, *The effect of infecting curves on knot concordance*, Int. Math. Res. Not. **2013** (2013), no. 1, 184–217.
24. S. Friedl, *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, Algebr. Geom. Topol. **4** (2004), 893–934.
25. J. Hillman, *Algebraic invariants of links*, 2nd ed., Ser. Knots Everything, vol. 52, World Scientific, Singapore, 2012.
26. B.-J. Jiang, *A simple proof that the concordance group of algebraically slice knots is infinitely generated*, Proc. Am. Math. Soc. **83** (1981), 189–192.
27. C. Kearton, *Blanchfield duality and simple knots*, Trans. Amer. Math. Soc. **202** (1975), 141–160.
28. J. P. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.
29. J. Levine, *Knot modules. I*, Trans. Amer. Math. Soc. **229** (1977), 1–50.
30. C. Livingston, *New examples of non-slice, algebraically slice knots*, Proc. Am. Math. Soc. **130** (2002), no. 5, 1551–1555.
31. C. Livingston, *Seifert forms and concordance*, Geom. Topol. **6** (2002), 403–408.
32. C. Livingston, *The algebraic concordance order of a knot*, J. Knot Theory Ramifications **19** (2010), no. 12, 1693–1711.
33. C. Livingston, *Knots which are not concordant to their reverses*, Q. J. Math., Oxf. II. Ser. **34** (1983), 323–328.
34. A. N. Miller and M. Powell, *Stabilization distance between surfaces*, Enseign. Math. (2) **65** (2019), no. 3–4, 397–440.
35. P. S. Ozsváth, A. I. Stipsicz, and Z. Szabó, *Concordance homomorphisms from knot Floer homology*, Adv. Math. **315** (2017), 366–426.
36. M. Powell, *A second order algebraic knot concordance group*, Algebr. Geom. Topol. **12** (2012), no. 2, 685–751.
37. M. Powell, *Twisted Blanchfield pairings and symmetric chain complexes*, Q. J. Math. **67** (2016), no. 4, 715–742.
38. M. Powell, *The four-genus of a link, Levine-Tristram signatures and satellites*, J. Knot Theory Ramifications **26** (2017), no. 2, 1740008, 28.
39. A. Ranicki, *High-dimensional knot theory*, Algebraic surgery in codimension 2, Springer Monographs in Mathematics, Springer, New York, 1998. (With an appendix by Elmar Winkelnkemper.)
40. T. Sakai, *Polynomials of invertible knots*, Math. Ann. **266** (1983), no. 2, 229–232.
41. M. Sakuma, *On strongly invertible knots*, Algebraic and topological theories. Papers from the symposium dedicated to the memory of Dr. Takehiko Miyata held in Kinosaki, October 30–November 9, 1984, 1986, pp. 176–196.
42. R. Schultz, *Homotopy invariants and G-manifolds: a look at the past fifteen years*, Group actions on manifolds (Boulder, Colo., 1983), 1985, p. 17–81.
43. P. A. Smith, *Transformations of finite period.*, Ann. Math. (2) **39** (1938), 127–164.
44. P. A. Smith, *Transformations of finite period. II*, Ann. Math. (2) **40** (1939), 690–711.
45. N. W. Stoltzfus, *Unraveling the integral knot concordance group*, Mem. Am. Math. Soc., vol. 192, American Mathematical Society (AMS), Providence, RI, 1977.
46. H. F. Trotter, *Knot module and Seifert matrices*, Knot theory (Proc. Sem., Plans-sur-Bex, 1977), 1978, pp. 291–299.